Journal of Basics and Applied Sciences Research (JOBASR)
ISSN (print): 3026-9091, ISSN (online): 1597-9962
Volume 3(6) November 2025 Special Issue
DOI: https://dx.doi.org/10.4314/jobasr.v3i6.13s

On the Efficiency of the Regression-Cum-Exponential Type Neutrosophic Estimators of Population Median in the Presence of Auxiliary Information

Magdalene Dooyum Gongul[1], Ahmed Audu[2*], Dauran N.S.[3] & Abdulhakeem Abdulazeez[4]
[1,2]Department of Statistics, Usmanu Danfodiyo University Sokoto, Nigeria
[3,4]Department of Computer Science Usmanu Danfodiyo University Sokoto, Nigeria
[*]Corresponding Author Email: ahmed.audu@udusok.edu.ng

ABSTRACT
Estimation of population median accurately is often challenging when data are vague, imprecise, or indeterminate. Classical estimators which rely on precise data, may produce biased and inefficient results under such conditions. neutrosophic estimators have been developed to address the issues of vagueness, impreciseness, or indeterminateness on median estimation. However, some recent efficient existing median estimators depend on unknown constants which makes them impracticable in real life situations unless if the unknown parameters are estimated using a sample which require huge resources. These existing median estimators are also ratio-based which are less efficient when the correlation between the study variable and auxiliary variable is negative. To address these problems, this study introduced regression-cum-exponential-type neutrosophic estimators for the population median which are efficient and free of unknown constants. The proposed median estimators are regression-base estimators which are efficient for both negative and positive correlation. Theoretical expressions for biases and mean squared errors (MSEs) of the proposed neutrosophic median estimators were derived up to the first order of approximation and the theoretical efficiency conditions over the related existing estimators were established. The performances of the proposed estimators were evaluated empirically using biases, MSEs and percent relative efficiency (PRE) through simulation studies. The results indicate that the proposed estimators consistently outperform classical and existing methods by achieving lower MSE and higher PRE with exception of few cases. The findings highlight the accuracy, efficiency, and practical applicability of the proposed neutrosophic approach for median estimation in uncertain environments.

Keywords:
Neutrosophic statistics, Population median, Auxiliary information, Bias, Mean square error (MSE), Percent Relative Efficiency (PRE).

INTRODUCTION

The main purpose of sampling theory is to enhance the accuracy of estimating unknown population parameter for a study variable on the basis of the auxiliary information. This approach is more effective when there is a strong correlation between the study variable and auxiliary variable. Common methods used to estimate population parameters include ratio, product and regression methods. The ratio method of estimation was first suggested by Cochran (1940) and can be used when there is a strong positive correlation between the study variable and the auxiliary variable. The product estimator was proposed by Murthy (1964) and is applied when there is a strong negative correlation, while the regression estimator can be applied when there is either negative or positive correlation and is generally more efficient than the ratio or product methods.

Several authors like Singh and Audu (2015), Audu et al. (2020), Singh et al. (2020), Audu et al. (2023) and Sher et al. (2025) have worked extensively in this direction

Auxiliary information, which refers to information used to improve the performance of estimator, was first utilized by Cochran (1940). Subsequent studies include Singh and Singh (1998), who developed an almost unbiased ratio and product-type estimator; Abu-Dayyeh *et al*. (2003), who extended estimators for more than two auxiliary variables; Singh *et al*. (2014), who constructed a ratio-type estimator using two auxiliary variables; Sharma and Singh (2014), who proposed a generalized median estimator; Lamichhane *et al.* (2017), who suggested an estimator for the finite population mean using the auxiliary variable's median; and Hussain et al. (2024), who showed that using auxiliary data reduces bias and mean square error (MSE) in median estimation.

How to cite this article: Magdalene D. G., Ahmed A., Dauran N.S. & Abdulhakeem A. (2025). On the Efficiency of the Regression-Cum-Exponential Type Neutrosophic Estimators of Population Median in the Presence of Auxiliary Information. *Journal of Basics and Applied Sciences Research*, 3(6)s, 107-125. https://dx.doi.org/10.4314/jobasr.v3i6.13s

Median estimators are used to determine the central location of a population and are valuable when data contain extreme values or are heavily skewed. Several studies have contributed to estimating the population median. Gross (1980) used the sample median in various sampling methods. Kuk and Mak (1989) proposed using the known median of an auxiliary variable. Rao et al. (1990), García and Cebrian (2001), Arcos *et al.* (2005), and Singh *et al.* (2007) proposed estimators for unknown population medians. Singh and Solanki (2013) developed classes of estimators using auxiliary information. Sharma and Singh (2015) proposed a family of estimators using auxiliary variables, while Muneer *et al*. (2020), Irfan *et al.* (2021), and Masood *et al.* (2024) developed efficient estimators for median using supplementary or robust auxiliary variables.

Classical statistical methods assume precise and crisp data. However, real-world data often exhibit vagueness, imprecision, and uncertainty. Zadeh (1965) introduced fuzzy statistics to manage uncertainty by assigning degrees of truth and falsehood, but it does not account for indeterminacy. Smarandache (1998) proposed Neutrosophic statistics to handle uncertainty, imprecision, vagueness, and incomplete information. Unlike classical or fuzzy methods, Neutrosophic methods allow for conflicting and indeterminate values, which is valuable for real-world data that are not fully reliable.

Neutrosophic methods have been applied in statistical estimation by Tahir *et al.* (2021), Vishwakarma and Singh (2023), Singh *et al.* (2024), and Masood et al. (2024), who developed robust estimators for population mean and median using auxiliary variables. Singh and Tiwari (2025) improved population mean estimators using unknown medians of two auxiliary variables. Singh *et al.* (2025) constructed an almost unbiased estimator for population median using neutrosophic information. However, their ratio-based estimator is less efficient when correlation between the study variable and auxiliary variable is negative.

Estimation of the population median plays an important role in Survey Sampling, particularly in skewed populations or when the presence of outliers may distort the usefulness of the mean. In real-world situations where data uncertainty, incompleteness, or indeterminacy exists, the neutrosophic framework provides a robust alternative to classical statistics by introducing truth, indeterminacy, and falsity components to handle such conditions.

Recent years have seen efforts to develop neutrosophic estimators for population parameters, though only a few have focused specifically on the population median using auxiliary information.

In Neutrosophic framework, observations often include quantitative data expressed as uncertain values within a range, typically denoted as (a,b), Smarandache (2014). There exist multiple representations for the interval form of a neutrosophic number, depending on the nature and degree of uncertainty involved. Singh *et al.* (2024) and Masood *et al.* (2024) describe neutrosophic interval value as $Z_N = Z_L + Z_U I_N$ where $I_N \in [I_L, I_U]$, the lower and upper values of the neutrosophic variables Z_N are denoted by Z_L and Z_U respectively, while I_N reflect the indeterminacy level in Z_N, with values from 0 to 1.

Let a population consist of $Q_N = (Q_{1N}, Q_{2N}, Q_{3N}, ..., Q_{NN})$. Each unit $Q_{iN} \in (i = 1, 2, ..., N)$ has two neutrosophic auxiliary variables and study variable $x_N \in [x_L, x_U]$, and $y_N \in [y_L, y_U]$. Let the sample of size $n_N \in [n_L, n_U]$ is chosen from Q_N. The sample and the population medians of the neutrosophic study and the auxiliary variables are represented by $\hat{M}_{yN}$ and $\hat{M}_{xN}$ and M_{yN}, and M_{xN} with probability density functions of $f_{yN}(M_{yN})$, and $f_{xN}(M_{xN})$ where $\hat{M}_{yN} \in (\hat{M}_{yL}, \hat{M}_{yU})$ and $\hat{M}_{xN} \in (\hat{M}_{xL}, \hat{M}_{xU})$, the correlation coefficient between the M_{yN} and M_{xN} is represented by ρ_{yxN} and is defined as;

$\rho_{yxN}(M_{yN}, M_{xN}) = (4p_{yxN}(y_N, x_N) - 1)$ where $p_{yxN}(y_N, x_N) = p_{yxN}(y_N \le M_{yN} \cap x_N \le M_{xN})$.

Review of Existing Population Median with Single Auxiliary Variable within Neutrosophic

Motivated by Gross (1980), Masood *et al.* (2024) proposed a neutrosophic traditional median estimator denoted by $\hat{M}_{0N}$ of population median as in (1.1). The variance of $\hat{M}_{0N}$ is given as in (1.2).

$$\hat{M}_{0N} = \hat{M}_{yN} \quad 1.1$$

$$\text{var}(\hat{M}_{0N}) = \lambda M_{yN}^2 C_{MyN}^2 \quad 1.2$$

Inspired by Kuk and Mak (1989), Masood *et al*. (2024) developed a novel neutrosophic traditional ratio estimator denoted by $\hat{M}_{RN}$ of population median as in (1.3). The bias and MSE of $\hat{M}_{RN}$ are given as in (1.4) and (1.5) respectively.

$$\hat{M}_{RN} = \hat{M}_{yN}\left(\frac{M_{xN}}{\hat{M}_{xN}}\right) \quad 1.3$$

$$Bias\left(\hat{M}_{RN}\right) \cong \lambda_N M_{yN}\left\{C_{MxN}^2 - C_{MyN}\right\} \quad 1.4$$

$$MSE\left(\hat{M}_{RN}\right) \cong \lambda_N M_{yN}^2\left\{C_{MyN}^2 + C_{MxN}^2 - 2C_{MyxN}\right\} \quad 1.5$$

However, the estimator $\hat{M}_{RN}$ performed better than $\hat{M}_{0N}$ if $\rho_{yxN} > 0.5\frac{C_{MxN}}{C_{MyN}}$

Using the concept of Bahl and Tuteja (1991), Masood *et al.* (2024) gave the neutrosophic exponential ratio-type estimator denoted by $\hat{M}_{EN}$ of population median as in (1.6). The bias and MSE of $\hat{M}_{EN}$ are given as in (1.7) and (1.8) respectively.

$$\hat{M}_{EN} = \hat{M}_{yN}\exp\left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}}\right) \quad 1.6$$

$$Bias\left(\hat{M}_{EN}\right) \cong M_{yN}\lambda_N\left(\frac{3}{8}C_{MxN}^2 - \frac{1}{2}C_{MyxN}\right) \quad 1.7$$

$$MSE\left(\hat{M}_{EN}\right) \cong M_{yN}^2\lambda_N\left(C_{MyN}^2 + \frac{1}{4}C_{MxN}^2 - C_{MyxN}\right) \quad 1.8$$

However, it has established that $\hat{M}_{EN}$ is more efficient than $\hat{M}_{0N}$ and $\hat{M}_{RN}$ if $\rho_{yxN} > 0.25\frac{C_{MxN}}{C_{MyN}}$ and $\rho_{yxN} < 0.75\frac{C_{MxN}}{C_{MyN}}$ respectively.

Masood *et al.* (2024) adapted difference estimator of population median and proposed neutrosophic difference estimator as given in (1.9)

$$\hat{M}_{D_{0N}} = \hat{M}_{yN} + d_{0N}\left(M_{xN} - \hat{M}_{xN}\right) \quad 1.9$$

At the optimal value of d_{0N} which is $d_{0N(opt)} = \frac{M_{yN}\rho_{yxN}C_{MyN}}{M_{xN}C_{MxN}}$, the minimum MSE of $\hat{M}_{D_{0N}}$, is given as in (1.10)

$$\text{var}\left(\hat{M}_{D_0}\right)_{\min} \cong M_{yN}^2 C_{MyN}^2 \lambda_N\left(1 - \rho_{yxN}^2\right) \quad 1.10$$

Adopted from Muneer *et al.* (2020), Masood *et al.* (2024) expressed the difference-type estimators dented by $\hat{M}_{D_{1N}}, \hat{M}_{D_{2N}}, \hat{M}_{D_{3N}}, \hat{M}_{D_{4N}}$ of population median as in (1.11), (1.12), (1.13), (1.14) respectively. The expressions for biases and MSEs of $\hat{M}_{D_{1N}}, \hat{M}_{D_{2N}}, \hat{M}_{D_{3N}}, \hat{M}_{D_{4N}}$ are as in (1.15) -(1.22).

$$\hat{M}_{D_{1N}} = d_{1N}\hat{M}_{yN} + d_{2N}(M_{xN} - \hat{M}_{xN}) \quad 1.11$$

$$\hat{M}_{D_{2N}} = \left\{d_{3N}\hat{M}_{yN} + d_{4N}(M_{xN} - \hat{M}_{xN})\right\}\left(\frac{M_{xN}}{\hat{M}_{xN}}\right), \quad 1.12$$

$$\hat{M}_{D_{3N}} = \left\{\begin{matrix} d_{5N}\hat{M}_{yN} + d_{6N} \\ \left(M_{xN} - \hat{M}_{xN}\right) \end{matrix}\right\}\exp\left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}}\right) \quad 1.13$$

$$\hat{M}_{D_{4N}} = \left\{\begin{matrix} d_{7N}\hat{M}_{yN} + d_{8N} \\ (M_{xN} - \hat{M}_{xN}) \end{matrix}\right\}\exp\left(\frac{M_{xN}}{M_{xN}} - 1\right) \quad 1.14$$

$$Bias(\hat{M}_{D_{1N}}) \cong (d_{1N} - 1)M_{yN}, \quad 1.15$$

$$Bias(\hat{M}_{D_{2N}}) \cong (d_{3N} - 1)M_{yN} + d_{3N}M_{yN}C_{1N} + d_{4N}M_{xN}B_{1N} \quad 1.16$$

$$Bias(\hat{M}_{D_{3N}}) \cong (d_{5N} - 1)M_{yN} + d_{5N}M_{yN}C_{2N} + d_{6N}M_{xN}B_{2N} \quad 1.17$$

$$Bias(\hat{M}_{D_{4N}}) \cong (d_{7N}-1)M_{yN} + d_{7N}M_{yN}C_{3N} + d_{8N}M_{xN}B_{3N} \quad 1.18$$

$$MSE(\hat{M}_{D_{1N}})_{\min} \cong \hat{M}_{yN}^2\left\{1-\frac{B_{0N}}{A_{0N}B_{0N}-C_{0N}^2+B_{0N}}\right\} \quad 1.19$$

$$MSE(\hat{M}_{D_{2N}})_{\min} \cong \hat{M}_{yN}^2\left\{1-\frac{\begin{pmatrix}A_{1N}B_{1N}^2+B_{1N}C_{1N}^2-2B_{1N}C_{1N}D_{1N}+\\ B_{1N}^2+2B_{1N}D_{1N}-2B_{1N}D_{1N}+B_{1N}\end{pmatrix}}{A_{1N}B_{1N}-D_{1N}^2+B_{1N}}\right\}, \quad 1.20$$

$$MSE(\hat{M}_{D_{3N}})_{\min} \cong \hat{M}_{yN}^2\left\{1-\frac{\begin{pmatrix}A_{2N}D_{2N}^2+B_{2N}C_{2N}^2-2C_{2N}D_{2N}E_{2N}\\ +2B_{2N}C_{2N}+D_{2N}^2-2D_{2N}E_{2N}+B_{2N}\end{pmatrix}}{A_{2N}B_{2N}-E_{2N}^2+B_{2N}}\right\}, \quad 1.21$$

$$MSE(\hat{M}_{D_{4N}})_{\min} \cong \hat{M}_{yN}^2\left\{1-\frac{\begin{pmatrix}A_{3N}D_{3N}^2+B_{3N}C_{3N}^2-2B_{3N}C_{3N}D_{3N}\\ +B_{3N}^2+2B_{3N}C_{3N}-2B_{3N}D_{2N}+B_{3N}\end{pmatrix}}{A_{3N}B_{3N}-D_{3N}^2+B_{3N}}\right\}. \quad 1.22$$

where $d_{iN}(i=1-8)$ are constants determined below by optimality considerations as

$$d_{1N(opt)}=\frac{B_{0N}}{A_{0N}B_{0N}-C_{0N}^2+B_{0N}}, d_{2N(opt)}=\frac{M_{yN}}{M_{xN}}\frac{B_{0N}}{A_{0N}B_{0N}-C_{0N}^2+B_{0N}},$$

$$d_{3N(opt)}=\frac{B_{1N}(C_{1N}-D_{1N}+1)}{A_{1N}B_{1N}-D_{1N}^2+B_{1N}}, d_{4N(opt)}=\frac{M_{yN}}{M_{xN}}\frac{(A_{1N}B_{1N}-C_{1N}D_{1N}+B_{1N}-D_{1N})}{(A_{1N}B_{1N}-D_{1N}^2+B_{1N})},$$

$$d_{7N(opt)}=\frac{B_{3N}(C_{3N}-D_{3N}+1)}{A_{3N}B_{3N}-D_{3N}^2+B_{3N})}, d_{8N(opt)}=\frac{M_{yN}}{M_{xN}}\frac{(A_{3N}B_{3N}-C_{3N}D_{3N}+B_{3N}-D_{3N})}{(A_{3N}B_{3N}-D_{3N}^2+B_{3N})},$$

$A_{0N}=\lambda_N C_{MyN}^2$, $B_{0N}=\lambda_N C_{MyN}^2$, $C_{0N}=\lambda_N C_{MyxN}$,
$A_{1N}=\lambda_N(C_{MyN}^2+3C_{MxN}^2-4C_{MyxN})$,
$B_{1N}=\lambda_N C_{MxN}^2$, $C_{1N}=\lambda_N(C_{MxN}^2-C_{MyxN})$,
$D_{1N}=\lambda_N(2C_{MxN}^2-C_{MyxN})$,
$A_{2N}=\lambda_N(C_{MyN}^2+C_{MxN}^2-2C_{MyxN})$,
$B_{2N}=\lambda_N C_{MxN}^2$, $C_{2N}=\lambda_N(\frac{3}{8}C_{MxN}^2-\frac{1}{2}C_{MyxN})$,
$D_{2N}=\lambda_N C_{MxN}^2/2$, $E_{2N}=\lambda_N(C_{MxN}^2-C_{MyxN})$,
$A_{3N}=\lambda_N(C_{MyN}^2+4C_{MxN}^2-4C_{MyxN})$,
$B_{3N}=\lambda_N C_{MxN}^2$, $C_{3N}=\lambda_N(\frac{3}{2}C_{MxN}^2-C_{MyxN})$, and
$D_{3N}=\lambda_N(2C_{MxN}^2-C_{MyxN})$.

Motivated by Irfan *et al.* (2021), Masood *et al.* (2024) developed a neutrosophic generalized ratio-type estimator represented by $T_{i(d)N}$ of finite population median as in (1.23). The bias and MSE of $T_{i(d)N}$ is given in (1.24) and (1.25) respectively.

$$T_{i(d)N}=\hat{M}_{yN}\left[\left\{m_{1N}\left(\frac{\psi_N\hat{M}_{xN}+\delta_N}{\psi_N M_{xN}+\delta_N}\right)^{\alpha_3}\right\}+\exp\left(\frac{M_{xN}-\hat{M}_{xN}}{M_{xN}+\hat{M}_{xN}}\right)\left\{m_{2N}\left(\frac{\psi_N M_{x1N}+\delta_N}{\psi\hat{M}_{xN}+\delta_N}\right)\right\}\right] \quad 1.23$$

$$Bias(T_{i(d)N}) = M_{yN}\left[m_{1N}\left(\begin{array}{l}1+\frac{\lambda_N C_{MxN}^2}{2}\left(\frac{3}{4}-\alpha_3\theta_N+\alpha_3(\alpha_3-1)\theta_N^2\right)\\ +\lambda_N\rho_N C_{MyN}C_{MxN}\left(\alpha_3\theta_N-\frac{1}{2}\right)\\ +m_{2N}\left(\begin{array}{l}1+\frac{\alpha_4(\alpha_4+1)\theta_N^2\lambda_N C_{MxN}^2}{2}\\ -\alpha_4\theta_N\lambda_N\rho_N C_{MyN}C_{MxN}\end{array}\right)-1\end{array}\right)\right] \quad 1.24$$

$$MSE(T_{i(d)N})_{\min} \cong M_{yN}^2\left[1-\frac{(A_{2N}A_{4N}^2+A_{1N}A_{5N}^2-2A_{3N}A_{4N}A_{5N})}{(A_{1N}A_{2N}-A_{3N}^2)}\right] \quad 1.25$$

Inspired by Sharma and Singh (2014), Singh *et al.* (2025) introduced the neutrosophic exponential estimator denoted by t_{1N} of population median as shown in (1.26). The bias and MSE of t_{1N} are given as in (1.27) and (1.28) respectively.

$$t_{1N} = \hat{M}_{yN}\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)^a \exp\left(\frac{b(\hat{M}_{xN}-M_{xN})}{\hat{M}_{xN}+M_{xN}}\right) \quad 1.26$$

$$Bias(t_{1N}) = M_{yN}\lambda_N\left[\begin{array}{l}\left(\left(\frac{a(a-1)}{2}\right)+\frac{ab}{2}-\frac{b}{4}+\frac{b^2}{8}\right)\\ C_{MxN}^2+aC_{MyxN}\end{array}\right] \quad 1.27$$

$$MSE(t_{1N}) = M_{yN}^2\lambda_N\left[\begin{array}{l}C_{MyN}^2+\left(a+\frac{b}{2}\right)^2 C_{MxN}^2\\ +2\left(a+\frac{b}{2}\right)C_{MyxN}\end{array}\right] \quad 1.28$$

Motivated by Mishra *et al.* (2017), Singh *et al.* (2025) proposed neutrosophic log type estimator expressed by t_{2N} of the population median as given in (1.29). The bias and MSE of t_{2N} are given as in (1.30) and (1.31) respectively.

$$t_{2N} = \hat{M}_{yN}\left(1+\log\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)\right) \quad 1.29$$

$$Bias(t_{2N}) = M_{yN}\left[\lambda_N C_{MyxN}-\frac{1}{2}C_{MxN}^2\right] \quad 1.30$$

$$MSE(t_{2N}) = M_{yN}^2\lambda_N\left[C_{MyN}^2+C_{MxN}^2+2C_{MyxN}^2\right] \quad 1.31$$

Using the concept of Gross (1980), Sharma and Singh (2014) and Mishra *et al.* (2017), Singh *et al.* (2025) proposed improved neutrosophic estimator denoted by t_{hN} for estimating the neutrosophic finite population median as in (1.32). The bias and MSE of the estimator t_{hN} are given as in (1.33) and (1.34) respectively.

$$t_{hN} = \alpha_{0N}\hat{M}_{yN}+\alpha_{1N}\hat{M}_{yN}\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)^a \exp\left(\frac{b(\hat{M}_{xN}-M_{xN})}{\hat{M}_{xN}+M_{xN}}\right)+ \alpha_{2N}\hat{M}_{yN}\left(1+\log\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)\right) \quad 1.32$$

$$Bias(t_{hN}) = M_{yN}\left[\begin{array}{l}\left(\begin{array}{l}\frac{a(a-1)}{2!}\alpha_{1N}+\frac{ab}{2}\alpha_{1N}\\ -\frac{b}{4}\alpha_{1N}+\frac{b^2}{8}\alpha_{1N}-\frac{1}{2}\alpha_{2N}\end{array}\right)\\ \lambda_N C_{MxN}^2+H_N\lambda_N C_{MyxN}\end{array}\right] \quad 1.33$$

$$MSE(t_{hN}) = M_{yN}^2\lambda_N\left[\begin{array}{l}C_{MyN}^2+H^2C_{MxN}^2+\\ 2H_N C_{MyxN}\end{array}\right] \quad 1.34$$

$$Min.MSE(t_{hN}) = M_{yN}^{2}\lambda_N C_{MyN}^{2}\left(1-\rho_{yxN}^{2}\right). \qquad 1.35$$

Several estimators have been suggested by Singh *et al.* (2025) and are shown to be efficient. However, their median estimators depend on unknown constants which make it impracticable in real life situation, also, their estimator is a ratio base estimator which is less efficient when the correlation between the study and auxiliary variable is negative. To address the above flaws in Singh *et al*. (2025) prompted the current study.

MATERIALS AND METHODS

Proposed Population Median Estimators

Building upon the findings of Singh et al. (2025) and addressing the gaps observed in their study, we have developed the following estimators.

$$T_{P1j} = \left[\hat{M}_{yN} + b_N\left(M_{xN} - \hat{M}_{xN}\right)\right] \exp\left(\frac{\left(k_i M_{xN} + l_i\right) - \left(k_i \hat{M}_{xN} + l_i\right)}{\left(k_i M_{xN} + l_i\right) + \left(k_i \hat{M}_{xN} + l_i\right)}\right) \qquad 1.36$$

$$T_{P2j} = \left[\hat{M}_{yN} + b_N\left(M_{xN} - \hat{M}_{xN}\right)\right] \left(\frac{k_i M_{xN} + l_i}{k_i \hat{M}_{xN}^{*} + l_i}\right) \qquad 1.37$$

$$T_{P3j} = \left[\hat{M}_{yN} + b_N\left(M_{xN} - \hat{M}_{xN}\right)\right] \left(\frac{k_i M_{xN} + l_i}{k_i \hat{M}_{xN} + l_i}\right) \qquad 1.38$$

$$T_{P4j} = \left[\hat{M}_{yN} + b_N\left(M_{xN} - \hat{M}_{xN}\right)\right] \exp\left(\frac{\left(k_i \hat{M}_{xN}^{*} + l_i\right) - \left(k_i \hat{M}_{xN} + l_i\right)}{\left(k_i M_{xN} + l_i\right) + \left(k_i \hat{M}_{xN} + l_i\right)}\right) \qquad 1.39$$

$$T_{P5j} = \left[\hat{M}_{yN} + b_N\left(M_{xN} - \hat{M}_{xN}\right)\right] \left(\frac{k_i \hat{M}_{xN}^{*} + l_i}{k_i M_{xN} + l_i}\right) \qquad 1.40$$

Where $M_{xN}^{*} = \dfrac{NM_{xN} - n\hat{M}_{xN}}{N-n}$, $b_N = \dfrac{\rho_{yxN} S_{yN}}{S_{xN}}$.

k and l are the parameters of the auxiliary variables x_N or can assume values 0, 1 and $k \neq 0$

Procedure for Deriving the Properties (Biases and MSEs) of the Proposed New Estimators

The procedures for deriving the properties of the proposed estimators were presented. And to obtain the Biases and MSEs of the proposed estimators, the below errors terms are defined.

$$e_{0N} = \frac{\hat{M}_{yN} - M_{yN}}{M_{yN}}, e_{1N} = \frac{\hat{M}_{xN} - M_{xN}}{M_{xN}},$$

$$e_{0N} \in \left[e_{0L}, e_{0U}\right], e_{1N} \in \left[e_{1L}, e_{1U}\right], E(e_{0N}) = 0,$$

$$E(e_{1N}) = 0,\ E(e_{0N}^{2}) = \lambda_N C_{MyN}^{2}, E(e_{1N}^{2}) = \lambda_N C_{MxN}^{2},$$

$$E(e_{0N}e_{1N}) = \lambda_N C_{MyxN}.$$

$$Bias(T_i) = E\left(T_i - M_{yN}\right) \quad i = 1,2,3,4,5 \qquad 1.41$$

$$MSE\left(T_i\right) = E\left(T_i - M_{yN}\right)^2 \ i = 1,2,3,4,5 \qquad 1.42$$

The biases and MSEs of estimators $T_{p_{1j}}, T_{p_{2j}}, T_{p_{3j}}, T_{p_{4j}}, T_{p_{5j}}$ are obtained by using the results of the expected values of the errors above.

Properties of the Proposed Estimators

This section derived and presented the biases and mean squared errors of the proposed estimators.

Bias and MSE of the Proposed Estimator T_{P1j}

Express T_{P1j} in terms of e_{0N} and e_{1N}, equation (2.1) is obtained

$$T_{P1j} = \left[\begin{array}{l} M_{yN}\left(1+e_{0N}\right) + \\ b_N\left(M_{xN} - M_{xN}\left(1+e_{1N}\right)\right)\end{array}\right] \exp\left[\frac{\left(k_i M_{xN} + l_i\right) - \left(k_i M_{xN}\left(1+e_{1N}\right) + l_i\right)}{\left(k_i M_{xN} + l_i\right) + \left(k_i M_{xN}\left(1+e_{1N}\right) + l_i\right)}\right] \qquad 2.1$$

Simplifying equation (2.1), equation (2.2) is obtained.

$$T_{P1j} = \left[M_{yN}\left(1+e_{0N}\right)-b_N M_{xN}e_{1N}\right] \exp\left[\frac{-k_i M_{xN}e_{1N}}{2\left(k_i M_{xN}+l_i\right)+k_i M_{xN}e_{1N}}\right] \quad 2.2$$

Let $\theta_i = \dfrac{k_i M_{xN}}{k_i M_{xN}+l_i}$

$$T_{P1j} = \left[M_{yN}\left(1+e_{0N}\right)-b_N M_{xN}e_{1N}\right] \exp\left[-\frac{\theta_i}{2}e_{1N}\left(1+\frac{\theta_i}{2}e_{1N}\right)^{-1}\right] \quad 2.3$$

$$T_{P1j} = \left[M_{yN}\left(1+e_{0N}\right)-b_N M_{xN}e_{1N}\right] \exp\left[-\frac{\theta_i}{2}e_{1N}+\frac{\theta_i^2}{4}e_{1N}^2\right] \quad 2.4$$

Subtracting M_{yN} from both sides of (2.4), equation (2.5) is obtained.

$$T_{P1j}-M_{yN} = M_{yN}e_{0N}-\left(M_{yN}\frac{\theta_i}{2}+M_{xN}b_N\right)e_{1N} +\left(\frac{3}{8}M_{yN}\theta_i^2-b_N\frac{\theta_i}{2}M_{xN}\right)e_{1N}^2 - M_{yN}\frac{\theta_i}{2}e_{0N}e_{1N} \quad 2.5$$

Taking expectation of both sides of equation (2.5) as in equation (2.6) and the bias is obtained as in equation (2.7)

$$E\left(T_{P1j}-M_{yN}\right) = \left(\frac{3}{8}M_{yN}\theta_i^2-\frac{b_N\theta_N}{2}M_{xN}\right) E\left(e_{1N}^2\right)-M_{yN}\frac{\theta_i}{2}E\left(e_{0N}e_{1N}\right) \quad 2.6$$

$$Bias\left(T_{P1j}\right) = \lambda_N\left[\left(\frac{3}{8}M_{yN}\theta_i^2-\frac{b_N\theta_i}{2}M_{xN}\right) C_{MxN}^2 - M_{yN}\frac{\theta_i}{2}C_{MyxN}\right] \quad 2.7$$

Squaring both sides of equation (2.5) of the first order of approximation take Expectation of both sides and obtain MSE as in (2.9)

$$\left(T_{P1j}-M_{yN}\right)^2 = \left[M_{yN}e_{0N}-\left(M_{yN}\frac{\theta_i}{2}+M_{xN}b_N\right)e_{1N}+\left(\frac{3}{8}M_{yN}\theta_i^2-b_N\frac{\theta_i}{2}M_{xN}\right)e_{1N}^2 - M_{yN}\frac{\theta_i}{2}e_{0N}e_{1N}\right]^2 \quad 2.8$$

$$MSE\left(T_{P1j}\right) = \lambda_N\left[M_{yN}^2 C_{MyN}^2 + \left(M_{yN}\frac{\theta_i}{2}+M_{xN}b_N\right)^2 C_{MxN}^2 - 2M_{yN}\left(\frac{\theta_i}{2}M_{yN}+M_{xN}b_N\right)C_{MyxN}\right] \quad 2.9$$

Bias and MSE of the Proposed Estimator T_{P2j}

Express T_{P2j} in terms of e_{0N} and e_{1N}, equation (2.10) is derived

$$T_{P2j} = \left[M_{yN}\left(1+e_{0N}\right)+b_N\left(M_{xN}\left(1+e_{1N}\right)\right)\right] \left(\frac{k_i M_{xN}+l_i}{k_i M_{xN}\left(1-\frac{n}{N-n}e_{1N}\right)+l_i}\right) \quad 2.10$$

Simplifying equation (2.10), (2.11) is obtained.

$$T_{P2j} = \left[M_{yN}\left(1+e_{0N}\right)-M_{xN}b_Ne_{1N}\right]\left(\frac{k_iM_{xN}+l_i}{k_iM_{xN}+l_i\left(1-M_{xN}\frac{n}{(N-n)\left(k_iM_{xN}+l_i\right)}e_{1N}\right)+l_i}\right) \quad 2.11$$

$$T_{P2j} = \left[M_{yN}+M_{yN}e_{0N}-M_{xN}b_Ne_{1N}\right]\left(1-\frac{\theta_i n}{N-n}e_{1N}\right)^{-1} \quad 2.12$$

$$T_{P2j} = \left[M_{yN}+M_{yN}e_{0N}-M_{xN}b_Ne_{1N}\right]\left(1+\theta_i\frac{n}{N-n}e_{1N}+\theta_i^2\left(\frac{n}{N-n}\right)^2e_{1N}^2\right) \quad 2.13$$

Subtracting M_{yN} from both sides of (2.13), (2.14) is obtained.

$$T_{P2j}-M_{yN} = \left\{\begin{array}{l} M_{yN}e_{0N}+ \\ \left(M_{yN}\theta_i\frac{n}{N-n}-M_{xN}b_N\right)e_{1N}+ \\ \left(\begin{array}{l} M_{yN}\theta_i^2\left(\frac{n}{N-n}\right)^2- \\ M_{xN}b_N\theta_i\frac{n}{N-n}\end{array}\right)e_{1N}^2+ \\ M_{yN}\theta_i\frac{n}{N-n}e_{0N}e_{1N}\end{array}\right\} \quad 2.14$$

Taking expectation of both sides of equation (2.14) as in equation (2.15) and the bias is obtained as in equation (2.16).

$$E\left(T_{P2j}-M_{yN}\right) = \left(\begin{array}{l} M_{yN}\theta_i^2\left(\frac{n}{N-n}\right)^2- \\ M_{xN}b_N\theta_i\frac{n}{N-n}\end{array}\right)E\left(e_{1N}^2\right)+ M_{yN}\theta_i\frac{n}{N-n}E\left(e_{0N}e_{1N}\right) \quad 2.15$$

$$Bias\left(T_{P2j}\right) = \lambda_N\left[\begin{array}{l}\left(\begin{array}{l} M_{yN}\theta_i\left(\frac{n}{N-n}\right)^2- \\ M_{xN}b_N\theta_i\frac{n}{N-n}\end{array}\right)C_{MxN}^2+ \\ M_{yN}\theta_i\frac{n}{N-n}C_{MyxN}\end{array}\right] \quad 2.16$$

Squaring both sides of equation (2.14) of the first order of approximation take Expectation of both sides and obtain MSE as in (2.18)

$$\left(T_{P2j}-M_{yN}\right)^2 = \left(\begin{array}{l} M_{yN}e_{0N}+ \\ \left(M_{yN}\theta_i\frac{n}{N-n}-M_{xN}b_N\right)e_{1N}+ \\ \left(M_{yN}\theta_i^2\left(\frac{n}{N-n}\right)^2-M_{xN}b_N\theta_i\frac{n}{N-n}\right)e_{1N}^2 \\ +M_{yN}\theta_i\frac{n}{N-n}e_{0N}e_{1N}\end{array}\right)^2 \quad 2.17$$

$$MSE\left(T_{P2j}\right) = \lambda_N\left[\begin{array}{l} M_{yN}^2C_{MyN}^2+\left(M_{yN}\theta_i\frac{n}{N-n}-M_{xN}b_N\right)^2 \\ C_{MxN}^2+ \\ 2M_{yN}\left(M_{yN}\theta_i\frac{n}{N-n}-M_{xN}b_N\right)C_{MyxN}\end{array}\right] \quad 2.18$$

Bias and MSE of the Proposed Estimator T_{P3j}

Express T_{P3j} in terms of e_{0N} and e_{1N}, equation (2.19) is derived

$$T_{P3j} = \left[M_{yN}\left(1+e_{0N}\right)+b_N\left(M_{xN}-M_{xN}\left(1+e_{1N}\right)\right)\right]\left(\frac{k_iM_{xN}+l_i}{k_iM_{xN}\left(1+e_{1N}\right)+l_i}\right) \quad 2.19$$

Simplifying equation (2.19), (2.22) is obtained.

$$T_{P3j} = \left[M_{yN}+M_{yN}e_{0N}-b_NM_{xN}e_{1N}\right]\left(\frac{k_iM_{xN}+l_i}{\left(k_iM_{xN}+l_i\right)\left(1+\dfrac{k_iM_{xN}e_{1N}}{k_iM_{xN}+l_i}\right)}\right) \quad 2.20$$

$$T_{P3j} = \left[M_{yN}+M_{yN}e_{0N}-b_NM_{xN}e_{1N}\right]\left(\frac{1}{1+\theta_ie_{1N}}\right) \quad 2.21$$

$$T_{P3j} = \left[M_{yN}+M_{yN}e_{0N}-b_NM_{xN}e_{1N}\right]\left(1-\theta_ie_{1N}+\theta_i^2e_{1N}^2\right) \quad 2.22$$

Subtracting M_{yN} from both sides of (2.22), (2.23) is obtained

$$T_{P3j}-M_{yN} = M_{yN}e_{0N}-\left(M_{yN}\theta_i+b_NM_{xN}\right)e_{1N} \left(M_{yN}\theta_i^2+M_{xN}b_N\theta_i\right)e_{1N}^2-M_{yN}\theta_ie_{0N}e_{1N} \quad 2.23$$

Taking expectation of both sides of equation (2.23) as in equation (2.24) and the bias is derived as in equation (2.25)

$$E\left(T_{P3j}-M_{yN}\right) = \left(M_{yN}\theta_i^2+M_{xN}b_N\theta_i\right)E\left(e_{1N}^2\right)-M_{yN}\theta_iE\left(e_{0N}e_{1N}\right) \quad 2.24$$

$$Bias\left(T_{P3j}\right) = \lambda_N\left[\left(M_{yN}\theta_i^2+M_{xN}b_N\theta_i\right)C_{MxN}^2 -M_{yN}\theta_iC_{MyxN}\right] \quad 2.25$$

Square both sides of equation (2.23) of the first order of approximation, take Expectation of both sides and obtain MSE as in (2.27)

$$\left(T_{P3j}-M_{yN}\right)^2 = M_{yN}e_{0N}^2+\left(M_{yN}\theta_i+b_NM_{xN}\right)^2e_{1N}^2 -2M_{yN}\left(M_{yN}\theta_i+b_NM_{xN}\right)e_{0N}e_{1N} \quad 2.26$$

$$MSE\left(T_{P3j}\right) = \lambda_N\left[M_{yN}^2C_{MyN}^2+\left(M_{yN}\theta_i+b_NM_{xN}\right)^2C_{MxN}^2 -2M_{yN}\left(M_{yN}\theta_i+b_NM_{xN}\right)C_{MyN}\right] \quad 2.27$$

Bias and MSE of the Proposed Estimator T_{P4j}

Express T_{P4j} in terms of e_{0N} and e_{1N}, equation (2.28) is obtained

$$T_{P4j} = \left[\left[M_{yN}\left(1+e_{0N}\right)+b_N\left(M_{xN}-M_{xN}\left(1+e_{1N}\right)\right)\right]\exp\left(\frac{\left(k_iM_{xN}\left(1-\dfrac{n}{N-n}e_{1N}\right)+l_i\right)-\left(k_iM_{xN}\left(1+e_{1N}\right)+l_i\right)}{\left(k_iM_{xN}\left(1-\dfrac{n}{N-n}e_{1N}\right)+l_i\right)+\left(k_iM_{xN}\left(1+e_{1N}\right)+l_i\right)}\right)\right] \quad 2.28$$

$$T_{P4j} = \left[M_{yN}\left(1+e_{0N}\right)-b_NM_{xN}e_{1N}\right]\exp\left(\frac{-k_iM_{xN}\left(\dfrac{n}{N-n}+1\right)e_{1N}}{2\left(k_iM_{xN}+l_i\right)-k_iM_{xN}\left(\dfrac{n}{N-n}-1\right)e_{1N}}\right) \quad 2.29$$

$$T_{P4j} = \left[M_{yN}\left(1+e_{0N}\right)-b_N M_{xN}e_{1N}\right] \exp\left(\frac{-\theta_i}{2}\left(\frac{n}{N-n}+1\right)e_{1N}\left(1-\frac{\theta_i}{2}\left(\frac{n}{N-n}-1\right)e_{1N}\right)^{-1}\right) \quad 2.30$$

$$T_{P4j} = \left[M_{yN}\left(1+e_{0N}\right)-b_N M_{xN}e_{1N}\right] \exp\left(\frac{\theta_i}{2}\left(\frac{n}{N-n}+1\right)e_{1N}+\frac{\theta_i^2}{4}\left(\frac{n}{N-n}-1\right)^2 e_{1N}^2\right) \quad 2.31$$

$$T_{P4j} = \left[M_{yN}+M_{yN}e_{0N}-b_N M_{xN}e_{1N}\right] \begin{pmatrix} 1+\frac{\theta_i}{2}\left(\frac{n}{N-n}+1\right)e_{1N}+ \\ \frac{\theta_i^2}{4}\left(\left(\frac{n}{N-n}\right)^2-1\right)e_{1N}^2+\frac{\left[\frac{\theta_i}{2}\left(\frac{n}{N-n}+1\right)e_{1N}\right]}{2!} \end{pmatrix} \quad 2.32$$

Subtracting M_{yN} from both sides of (2.32), (233) is obtained.

$$T_{P4j}-M_{yN} = \begin{bmatrix} M_{yN}\frac{\theta_i}{2}\left(\frac{n}{N-n}+1\right)e_{1N}+ \\ M_{yN}\frac{\theta_i^2 N\left(4n-N\right)}{8\left(N-n\right)}e_{1N}^2+M_{yN}e_{0N} \\ +M_{yN}\frac{\theta_i}{2}\left(\frac{n}{N-n}+1\right)e_{0N}e_{1N} \\ -b_N M_{xN}\frac{\theta_i}{2}\left(\frac{n}{N-n}+1\right)e_{1N}^2 \end{bmatrix} \quad 2.33$$

Taking expectation of both sides of equation (2.33) as shown in equation (2.34) and the bias is obtained as in equation (2.35).

$$E\left(T_{P4j}-M_{yN}\right) = \begin{bmatrix} \begin{Bmatrix} \frac{M_{yN}\theta_i^2 N\left(4n-N\right)}{8\left(N-n\right)^2}- \\ b_N M_{xN}\frac{\theta_i}{2}\left(\frac{n}{\left(N-n\right)}+1\right) \end{Bmatrix}\lambda_N C_{MxN}^2+ \\ M_{yN}\frac{\theta_i}{2}\left(\frac{n}{\left(N-n\right)}+1\right)\lambda_N C_{MyxN} \end{bmatrix} \quad 2.34$$

$$Bias\left(T_{P4j}\right) = \lambda_N \begin{bmatrix} \begin{Bmatrix} \frac{M_{yN}\theta_i^2 N\left(4n-N\right)}{8\left(N-n\right)^2}- \\ b_N M_{xN}\frac{\theta_i}{2}\left(\frac{n}{\left(N-n\right)}+1\right) \end{Bmatrix} C_{MxN}^2+ \\ M_{yN}\frac{\theta_i}{2}\left(\frac{n}{\left(N-n\right)}+1\right)C_{MyxN} \end{bmatrix} \quad 2.35$$

Squaring both sides of equation (2.33) of the first order of approximation, take Expectation of both sides and obtain MSE as in (2.36).

$$MSE\left(T_{P4j}\right) = \lambda_N \begin{bmatrix} M_{yN}^2 C_{MyN}^2+\left(M_{yN}\frac{\theta_i}{2}\left(\frac{n}{N-n}+1\right)-b_N M_{xN}\right)^2 C_{MxN}^2 \\ +2M_{yN}\left(M_{yN}\frac{\theta_i}{2}\left(\frac{n}{N-n}+1\right)-b_N M_{xN}\right)C_{MyxN} \end{bmatrix} \quad 2.36$$

Bias and MSE of the Proposed Estimator T_{P5j}

Express T_{P5j} in terms of e_{0N} and e_{1N}, equation (2.37) is derived

$$T_{P5j} = \left[M_{yN}\left(1+e_{0N}\right)+b_N\left(M_{xN}-M_{xN}\left(1+e_{1N}\right)\right)\right] \left(\frac{k_i M_{xN}\left(1-\frac{n}{N-n}e_{1N}\right)+l_i}{k_i M_{xN}+l_i}\right) \quad 2.37$$

Simplifying equation (2.37), (2.40) is obtained

$$T_{P5j} = \left[M_{yN}\left(1+e_{0N}\right)-b_N M_{xN}e_{1N}\right] \left(\frac{k_i M_{xN}+li-M_{xN}\frac{n}{N-n}e_{1N}}{k_i M_{xN}+l_i}\right) \quad 2.38$$

$$T_{P5j} = \left[M_{yN}\left(1+e_{0N}\right)-b_N M_{xN}e_{1N}\right]\left(\frac{k_i M_{xN}+l_i\left(1-M_{xN}\frac{n}{(N-n)(k_i M_{xN}+l_i)}e_{1N}\right)}{k_i M_{xN}+l_i}\right) \quad 2.39$$

$$T_{P5j} = \left[M_{yN}+M_{yN}e_{0N}-b_N M_{xN}e_{1N}\right]\left(1-\frac{\theta_i n}{N-n}e_{1N}\right) \quad 2.40$$

Subtracting M_{yN} from both sides of (2.40), (2.41) is obtained

$$T_{P5j}-M_{yN} = \left[\begin{array}{l} M_{yN}e_{0N}-\left(M_{yN}\theta_i\frac{n}{N-n}+b_N M_{xN}\right)e_{1N} \\ +b_N M_{xN}\theta_i\frac{n}{N-n}e_{1N}^2 - M_{yN}\theta_i\frac{n}{N-n}e_{0N}e_{1N}\end{array}\right] \quad 2.41$$

Taking expectation of both sides of equation (2.41) as in equation (2.42) and the bias is obtained as in equation (2.43)

$$E\left(T_{P5j}-M_{yN}\right)= b_N M_{xN}\theta_i\frac{n}{N-n}E\left(e_{1N}^2\right)- M_{yN}\theta_i\frac{n}{N-n}E\left(e_{0N}e_{1N}\right) \quad 2.42$$

$$Bias\left(T_{P5j}\right)= \lambda_N\left[\begin{array}{l} b_N M_{xN}\theta_i\frac{n}{N-n}C_{MxN}^2 \\ -M_{yN}\theta_i\frac{n}{N-n}C_{MyxN}\end{array}\right] \quad 2.43$$

Square both sides of equation (2.41) of the first order of approximation, take Expectation of both sides and obtain MSE as in (2.45)

$$\left(T_{P5j}-M_{yN}\right)^2 = \left[\begin{array}{l} M_{yN}^2 e_{0N}^2+\left(M_{yN}\theta_i\frac{n}{N-n}+b_N M_{xN}\right)^2 e_{1N}^2 \\ -2M_{yN}\left(M_{yN}\theta_i\frac{n}{N-n}+b_N M_{xN}\right)e_{0N}e_{1N}\end{array}\right] \quad 2.44$$

$$MSE\left(T_{P5j}\right)= \lambda_N\left[\begin{array}{l} M_{yN}^2 C_{MyN}^2 + \\ \left(M_{yN}\theta_i\frac{n}{N-n}+b_N M_{xN}\right)^2 C_{MxN}^2 \\ -2M_{yN}\left(M_{yN}\theta_i\frac{n}{N-n}+b_N M_{xN}\right)C_{MyxN}\end{array}\right] \quad 2.45$$

RESULTS AND DISCUSSION

Empirical Study for Efficiency Comparison

In this section, a simulation study was conducted to evaluate the performance of the proposed estimators. A population of 1,000 units was generated and samples of sizes 50, 100, 150, 200, and 250 were drawn 1,000 times using Simple Random Sampling Without Replacement (SRSWOR). The Bias, Mean Squared Error (MSE), and Percent Relative Efficiency (PRE) of the estimators were calculated using equations (2.46), (2.47), and (2.48), respectively, providing a basis for comparing their accuracy and efficiency.

$$Bias\left(T\right)=\frac{1}{1000}\sum_{j=1}^{1000}\left(T-\bar{Y}\right) \quad 2.46$$

$$MSE\left(T\right)=\frac{1}{1000}\sum_{i=1}^{1000}\left(T-\bar{Y}\right)^2 \quad 2.47$$

$$PRE\left(T\right)=\left(\frac{MSE\left(t_{0N}\right)}{MSE\left(T\right)}\right)\times 100 \quad 2.48$$

Simulation procedure is described in the steps below;

Step 1: Population of size $N=1000$ for variable $X\ and\ Y$ are generated using Neutrosophic function defined in R Package

Step 2: Compute parameters of auxiliary and study variables from $X\ and\ Y$

Step 3: Take a random sample of size n from population generated in step 1

Step 4: Compute Biases, MSEs and PREs for each estimator using (2.46), (2.47) and (2.48) respectively

Step 5: Repeat step 3 and 4, 1000 times

Step 6: Compute the averages of results of step 4

Step 7: Display the results of step 6

Table 1: Biases, MSEs and PREs of the Proposed and Existing Estimators for n=50

Estimators	Biases		MSEs		PREs	
	T-Values	F-Values	T-Values	F-Values	T-Values	F-Values
t_{0N}	-0.2508	-0.4231	9.8165	39.4340	100	100
t_{1N}	0.1976	-0.0746	54.6121	185.0399	18.4667	20.80
t_{2N}	-0.1927	0.1069	1.9490	2.0393	592.6783	1843.88
t_{3N}	-0.0891	0.3553	27.3784	98.3400	36.4847	39.58
t_{4N}	-0.2872	-0.2732	1.2647	6.9556	724.7424	572.37
t_{5N}	0.6090	0.4197	92.2469	300.9537	11.0102	11.01
t_{6N}	0.0314	0.7146	12.7838	27.9017	86.4410	145.51
t_{7N}	-0.0533	-0.5069	54.2892	185.337	18.6463	21.04
t_{1hN}	-3.1429	78.8538	686.3454	6231.401	1.5287	0.54
t_{2hN}	0.4148	0.6645	352.0659	2.1453	1.5287	1779.51
Members of the Proposed Estimators						
T_{P11j}	-0.1655	0.4341	4.3178	14.4796	224.163	272.34
T_{P12j}	0.12417	0.2343	4.6246	1.8336	212.267	2150.63
T_{P13j}	0.0745	0.1578	1.9387	8.8228	506.344	446.96
T_{P14j}	1.59318	0.4667	7.1055	2.6087	138.153	1511.63
T_{P15j}	1.3549	0.14599	2.5756	1.3983	381.811	2820.14
T_{P21j}	-0.0116	0.1219	0.3583	0.5096	2738	7738.22
T_{P22j}	0.7145	0.0793	8.2796	1.0729	118.560	3675.46
T_{P23j}	1.6080	0.6462	3.0989	4.8474	316.774	813.51
T_{P24j}	0.8054	0.3617	2.6265	1.4697	373.748	2683.13
T_{P25j}	0.2428	0.7815	6.4267	7.5774	152.746	520.42
T_{P31j}	0.1219	1.2068	4.3182	14.4719	227.339	272.487
T_{P32j}	0.0716	0.9799	4.6244	1.8336	212.295	2150.633
T_{P33j}	-0.6081	0.6462	1.9385	8.8227	506.527	446.961
T_{P34j}	-0.8055	0.13589	7.1067	2.6087	138.144	1511.634
T_{P35j}	1.2661	0.7598	2.5761	1.3982	381.061	2820.340
T_{P41j}	-0.1614	-0.4554	4.7444	15.9145	206.907	247.779
T_{P42j}	-1.9463	1.0098	5.9334	10.997	165.445	358.589
T_{P43j}	0.3093	-0.9144	1.8111	6.4870	542.019	607.893
T_{P44j}	1.5246	0.59434	4.8339	3.8104	203.076	1034.904
T_{P45j}	0.5354	1.4382	2.5745	1.3097	381.297	3010.919
T_{P51j}	-0.3193	-0.0766	0.3369	0.6483	2913.773	6082.678
T_{P52j}	0.7143	0.9795	8.2782	1.0727	118.583	3676.144
T_{P53j}	1.6078	-0.4644	3.0982	4.8464	316.845	813.676
T_{P54j}	-0.0804	0.3618	2.6260	1.4694	373.819	2683.680
T_{P55j}	0.0724	0.7812	6.4278	7.5787	152.719	520.327

Table 2: Biases, MSEs and PREs of the Proposed and Existing Estimators for n=100

Estimators	Biases		MSEs		PREs	
	T-Values	F-Values	T-Values	F-Values	T-Values	F-Values
t_{0N}	0.2671	0.1121	4.8869	17.0762	100	100
t_{1N}	0.5425	0.2489	27.8963	83.8070	17.5179	20.3757
t_{2N}	0.2399	0.3558	1.0885	1.1741	449.053	1454.416
t_{3N}	0.3734	0.1324	13.8538	43.7219	35.2744	39.0565
t_{4N}	0.2228	0.1867	0.6738	3.0057	725.214	568.138
t_{5N}	0.7744	0.4613	47.3547	138.226	10.3197	12.3539
t_{6N}	0.3187	0.6196	6.1303	11.6057	79.7158	147.137
t_{7N}	0.4180	0.0581	27.3057	82.3396	17.8968	20.7388
t_{1hN}	-1.7109	80.0574	350.806	6414.322	1.3930	0.2662
t_{2hN}	0.3073	0.5863	168.479	0.9081	1.3930	1880.336
Members of the Proposed Estimators						
T_{P11j}	0.2396	0.4993	2.7375	7.0806	178.517	241.169
T_{P12j}	1.2046	0.2889	1.6930	1.7157	288.653	995.291
T_{P13j}	0.7572	1.9395	6.5320	7.4373	74.815	229.828
T_{P14j}	0.0364	1.9815	3.6847	4.7579	132.627	358.902
T_{P15j}	0.5236	3.7819	2.7433	1.6682	178.139	1023.630
T_{P21j}	0.2085	0.2483	0.3556	0.6044	1374.269	2825.314
T_{P22j}	2.7394	0.9944	7.5061	9.8909	65.106	172.646
T_{P23j}	1.6349	1.0596	2.6737	4.3520	182.777	392.376
T_{P24j}	0.8383	3.6736	2.3414	1.3496	208.717	1265.279
T_{P25j}	0.2774	1.0450	5.2975	6.4734	92.249	263.790
T_{P31j}	0.3461	0.8259	2.7375	7.0806	178.517	241.169
T_{P32j}	0.0274	0.0947	1.6930	1.7157	288.653	995.291
T_{P33j}	1.6349	0.6596	6.5320	7.4361	74.815	2289.639
T_{P34j}	0.8384	0.3646	3.6846	4.757	132.630	358.970
T_{P35j}	1.2980	0.08496	2.7433	1.6682	178.139	1023.630
T_{P41j}	0.2408	0.5195	3.2868	8.5576	148.683	199.544
T_{P42j}	-2.1234	-2.6344	5.1990	1.9901	93.996	858.115
T_{P43j}	1.7389	1.7943	5.9974	5.36	81.484	318.586
T_{P44j}	0.3153	1.6926	2.2297	2.66	219.173	641.962
T_{P45j}	0.2348	0.3719	2.7415	1.5458	178.256	1104.684
T_{P51j}	1.8433	0.2850	0.4154	0.8313	1176.432	2054.156
T_{P52j}	2.7388	1.9947	7.5032	9.8961	65.131	172.555
T_{P53j}	1.6343	0.6598	2.6723	4.3555	182.872	392.061
T_{P54j}	0.4836	0.3674	2.3403	1.3506	208.815	1264.342
T_{P55j}	0.7279	0.8024	5.2999	6.4692	92.207	263.962

Table 3: Biases, MSEs and PREs of the Proposed and Existing Estimators for n=150

Estimators	Biases		MSEs		PREs	
	T-Values	F-Values	T-Values	F-Values	T-Values	F-Values
t_{0N}	0.1408	-0.3856	2.8730	13.0227	100	100
t_{1N}	-0.4135	-0.0366	15.4559	59.8245	18.5884	21.7681
t_{2N}	-0.0023	0.1395	0.5110	0.7536	562.2494	1728.016
t_{3N}	0.2607	0.4087	7.8198	32.0887	36.7402	40.5834
t_{4N}	0.0532	0.2963	0.4607	2.5214	623.6183	516.4812
t_{5N}	0.5990	0.2586	25.9440	96. 3395	11.0738	13.5175
t_{6N}	-0.0257	0.0845	3.0072	8.1030	95.5352	160.7149
t_{7N}	0.3484	0.4981	15.1928	60.0401	18.9102	21.6900
t_{1hN}	-1.4468	82.7523	187.9962	6852.202	1.5282	0.1901
t_{2hN}	-0.4846	0.0022	89.6804	0.3996	1.5282	3258.949
Members of the Proposed Estimators						
T_{P11j}	-0.0299	-0.0062	1.3893	4.9761	206.795	262.232
T_{P12j}	1.9617	0.8310	4.2117	1.1018	68.215	1181.948
T_{P13j}	1.2124	0.4764	1.5905	3.5308	180.635	368.831
T_{P14j}	1.1069	0.3339	2.7834	1.4023	103.219	928.667
T_{P15j}	0.5521	0.5008	3.0500	2.7549	94.197	472.710
T_{P21j}	0.0455	-0.2386	0.3348	0.8468	858.124	1537.872
T_{P22j}	1.2906	-1070.385	8.4499	1.1458	34.000	1136.560
T_{P23j}	1.7709	0.7174	3.1367	5.1482	91.593	252.956
T_{P24j}	0.5231	0.4044	2.7371	1.6360	104.965	796.009
T_{P25j}	0.8103	1.9109	6.5686	8.2999	43.738	156.902
T_{P31j}	-0.0038	0.2655	1.3893	4.9761	206.794	261.705
T_{P32j}	1.2906	0.0107	4.2117	1.1018	68.214	818.883
T_{P33j}	1.7709	0.7174	1.5903	3.5308	180.658	368.831
T_{P34j}	0.5231	0.4017	2.7834	1.4023	103.219	928.667
T_{P35j}	0.8147	0.8895	3.0501	2.7549	94.193	472.710
T_{P41j}	-0.0403	0.0227	1.8533	6.7195	155.021	193.805
T_{P42j}	-2.3147	-2.8456	5.8186	3.2019	23.876	155.410
T_{P43j}	0.1185	0.4425	1.4659	2.5644	195.988	507.826
T_{P44j}	0.9578	1.6742	1.5149	2.4380	189.649	534.155
T_{P45j}	0.5519	0.4935	3.0466	2.5805	94.302	534.155
T_{P51j}	-0.0209	-0.1737	0.3305	0.9350	869.288	1392.802
T_{P52j}	0.1904	0.1070	8.4392	1.1468	34.044	1135.569
T_{P53j}	1.7691	0.7178	3.1309	5.1549	91.762	252.628
T_{P54j}	0.5226	0.4046	2.7327	1.6379	105.134	795.085
T_{P55j}	1.8110	0.9104	6.5786	8.2902	43.671	157.085

Table 4.: Biases, MSEs and PREs of the Proposed and Existing Estimators for n=200

Estimators	Biases		MSEs		PREs	
	T-Values	F-Values	T-Values	F-Values	T-Values	F-Values
t_{0N}	-0.1713	0.2746	2.2428	7.8007	100	100
1 t_{1N}	-0.2989	0.3933	11.8997	36.6061	18.847	21.310
t_{2N}	0.0617	0.3175	0.4799	0.6422	467.376	1214.657
t_{3N}	-0.2480	0.3140	6.1139	19.5032	36.683	39.997
t_{4N}	-0.0684	0.2755	0.3356	1.4801	668.325	527.059
t_{5N}	-0.3241	0.5125	19.5535	59.1311	11.469	13.192
t_{6N}	0.2186	0.4003	2.7910	5.4803	80.356	142.342
t_{7N}	-0.3511	0.3130	12.0591	36.6781	18.598	21.268
t_{1hN}	0.8411	84.1229	139.002	7078.983	1.6135	0.1102
t_{2hN}	1.0656	0.4159	76.2431	0.4607	1.6135	1693.221
Members of the Proposed Estimators						
T_{P11j}	0.1207	0.3580	1.2771	3.4277	175.617	227.578
T_{P12j}	0.2795	1.1535	8.3352	1.7584	26.906	443.625
T_{P13j}	1.6570	0.7252	2.9071	6.8506	77.149	113.869
T_{P14j}	0.28411	1.447075	1.2671	2.3082	177.003	337.956
T_{P15j}	0.5960	0.4002	3.5533	1.6893	63.119	461.771
T_{P21j}	-0.0664	0.2927	0.3532	0.8461	634.994	921.960
T_{P22j}	0.3048	0.1102	9.2959	1.2145	24.127	642.300
T_{P23j}	1.8802	0.7464	3.5358	5.5720	63.431	139.998
T_{P24j}	0.5582	0.4160	3.1172	1.7308	71.949	450.699
T_{P25j}	0.8791	0.9639	7.7304	9.2939	29.013	83.934
T_{P31j}	0.2908	0.4693	1.2771	3.4277	175.617	227.578
T_{P32j}	1.3050	0.1102	8.3352	1.7584	26.908	443.625
T_{P33j}	1.8802	0.7464	2.9071	6.8506	77.149	113.869
T_{P34j}	0.5582	0.4133	1.2671	2.3083	177.003	337.941
T_{P35j}	0.8733	0.9456	3.5533	1.6893	63.119	461.771
T_{P41j}	0.1519	0.3692	1.8861	5.0678	118.912	153.927
T_{P42j}	-2.259	-3.3620	5.6297	2.3701	39.839	329.130
T_{P43j}	1.6418	0.6794	2.7621	5.2465	81.199	148.684
T_{P44j}	2.5674	0.6638	8.3877	1.8015	26.739	433.011
T_{P45j}	0.5963	0.3948	3.5572	1.5998	63.050	487.605
T_{P51j}	0.0295	0.2922	0.4280	1.1136	524.019	700.494
T_{P52j}	0.3051	1.1020	9.3161	1.2147	24.074	642.191
T_{P53j}	0.1883	0.7464	3.5478	5.5739	63.217	139.950
T_{P54j}	0.5590	0.4160	3.1265	1.7314	71.735	450.543
T_{P55j}	1.8779	0.9638	7.7088	9.2913	29.094	83.957

Table 5: Biases, MSEs and PREs of the Proposed and Existing Estimators for n=250

Estimators	Biases		MSEs		PREs	
	T-Values	F-Values	T-Values	F-Values	T-Values	F-Values
t_{0N}	-0.2167	0.1911	1.6798	5.1639	100	100
t_{1N}	-0.4637	0.1669	8.9810	24.3993	18.704	21.164
t_{2N}	0.1115	0.3249	0.4036	0.5352	416.241	964.857
t_{3N}	-0.3500	0.1655	4.6115	12.9723	36.427	39.807
t_{4N}	-0.0630	0.2441	0.2486	0.9919	675.743	520.589
t_{5N}	-0.5576	0.1954	14.7273	39.4289	11.406	13.097
t_{6N}	0.3070	0.4334	2.2475	3.9192	74.743	131.760
t_{7N}	-0.5038	0.1125	9.1481	24.4950	18.362	21.081
t_{1hN}	1.5897	84.1148	105.4565	7076.842	1.5928	0.0730
t_{2hN}	1.4770	0.3769	105.4565	0.3684	1.5928	1401.637
Members of the Proposed Estimators						
T_{P11j}	0.1915	0.3886	1.0380	2.5020	161.830	206.391
T_{P12j}	0.00276	0.0108	7.8660	1.4344	21.355	360.004
T_{P13j}	0.1625	1.6850	2.7582	5.6409	60.902	91.544
T_{P14j}	2.5721	0.0974	9.2827	9.1568	18.096	56.394
T_{P15j}	0.0596	0.0394	3.5534	1.6098	47.273	320.779
T_{P21j}	-0.0857	0.2613	0.3953	0.8933	424.943	578.070
T_{P22j}	0.0511	0.0110	9.3101	1.2145	18.043	425.187
T_{P23j}	1.8819	1.7464	3.5421	5.5721	47.424	92.674
T_{P24j}	0.5587	0.0416	3.1223	1.7309	53.800	298.336
T_{P25j}	0.8809	0.9644	7.7625	9.3033	21.640	55.506
T_{P31j}	0.3972	0.5156	1.0380	2.5020	161.830	206.391
T_{P32j}	0.3533	1.1021	7.8660	1.4345	21.355	359.979
T_{P33j}	0.1882	0.7464	2.7582	5.6409	60.902	91.544
T_{P34j}	1.5586	0.4133	9.2827	9.1568	18.096	56.394
T_{P35j}	0.008717	0.0009	3.5534	1.6098	47.273	320.779
T_{P41j}	0.2481	0.4158	1.7026	4.0732	98.661	126.777
T_{P42j}	-2.2370	-3.5088	5.4692	21.627	30.713	23.877
T_{P43j}	1.6269	0.6520	2.6797	4.5351	62.686	113.865
T_{P44j}	2.3895	2.2097	6.5510	1.4740	25.642	350.332
T_{P45j}	0.0596	0.3906	3.5623	1.5459	47.155	334.038
T_{P51j}	0.1096	0.32756	0.5220	1.2887	321.801	400.706
T_{P52j}	0.0030	1.1024	9.3500	1.2156	17.966	424.803
T_{P53j}	0.0188	1.7469	3.5656	5.5802	47.111	92.540
T_{P54j}	0.0056	0.0416	3.1403	1.7332	53.492	297.940
T_{P55j}	1.8786	1.9639	7.7204	9.2919	21.758	55.574

Tables 1, 2, 3, 4 and 5 compares the performance of several estimators of the population median, including sample median, estimators of Singh *et al.* (2025) with that of the proposed neutrosophic estimators based on Bias, Mean Square Error (MSE), and Percent Relative Efficiency (PRE) under both Truth-values and False-values of neutrosophic data for sample of sizes 50, 100, 150, 200 and 250 respectively conditions. The results demonstrate that the proposed Neutrosophic estimators perform markedly better than the traditional estimators in terms of accuracy, stability, and overall efficiency.

In Table 1, in terms of Bias, the proposed estimators record relatively small and consistent values, ranging from –1.95 to 1.61, while the traditional estimators show larger and unstable values from –3.14 to 0.71 for T values and –0.51 to 78.85 for F-values. These fluctuations in the traditional estimators suggest a higher level of sensitivity to uncertainty, which reduces their reliability. A similar pattern is observed in the MSE results. The traditional estimators display large MSE values, between 1.26 and 686.35 for Truth values and 2.03 and 6231.40 for False values, implying low precision and greater estimation error. Conversely, the proposed Neutrosophic estimators maintain substantially lower MSE values ranging from 0.33 to 8.28 for T-values and 0.51 to 15.91 for F-values—indicating improved precision and reduced error in estimating the population median even under uncertain data conditions. The PRE results further reinforce the superiority of the proposed estimators. Whereas the traditional estimators mostly record efficiency values below 100, the proposed Neutrosophic estimators achieve considerably higher efficiencies, with PRE values ranging from 118.56 to 7738.22. One proposed estimator T_{P21j} attained PRE values of 2738.00 under the Truth values and 7738.22 under the False value, showing a remarkable increase in efficiency compared to the traditional estimators.

In Table 2, the proposed estimators generally have smaller biasess, with several very close to zero. Estimator T_{P21j} recorded 0.2085 under Truth-values and 0.2483 under False-values, while T_{P45j} showed 0.2348 and 0.3719, respectively. These results suggest that the proposed estimators produce estimates closer to the actual population median, demonstrating near-unbiased performance even under uncertain or indeterminate data conditions. The MSE values further illustrate the advantage of the proposed estimators. Existing methods show MSEs ranging from 0.67 to 47.35 under Truth-values and 3.01 to 138.23 under False-values, reflecting higher estimation errors. The proposed estimators on the other hand, achieve substantially smaller MSEs, typically between 0.35 and 7.51 under Truth-values and 0.60 and 9.90 under False-values. Estimator T_{P21j} attained an MSE of 0.3556 under Truth-values and 0.6044 under False-values, while T_{P51j} recorded 0.4154 and 0.8313, indicating greater accuracy and reliability. The PRE results reinforce these observations. While existing estimators exhibit PRE values between 17 and 725 under Truth-values and 20 to 568 under False-values, the proposed estimators achieve markedly higher efficiency, ranging from 65 to 1374 under Truth-values and 172 to 2825 under False-values. demonstrating significantly enhanced efficiency relative to conventional methods.

The findings in Table 3 indicate that the proposed estimators consistently surpass the traditional methods across all criteria. In terms of bias, their values range from approximately –0.0299 to 1.9617, showing that the estimates are very close to the true population median. Conversely, the conventional estimators exhibit higher and more variable bias, reflecting increased sensitivity to uncertainty and variability in the data. The MSE results further highlight the advantage of the proposed estimators. Existing estimators show a wide range of MSEs, from 0.4607 to 187.9962 under Truth-values and 2.5214 to 6852.202 under False-values. The reference estimator recorded MSEs of 2.8730 and 13.0227, while some conventional estimators showed extremely high values, indicating low reliability. By comparison, the proposed estimators consistently achieve smaller and more stable MSEs. For instance, one estimator recorded 1.3839 under Truth-values and 4.9761 under False-values, and another recorded 1.5903 and 3.5308, demonstrating substantial improvement in precision and consistency. The PRE results reinforce these observations. Traditional estimators produced PREs ranging from 11% to 624% under Truth-values and 13% to 1728% under False-values. However, the proposed estimators achieved significantly higher PREs, with some reaching 206.80% under Truth-values and 262.23% under False-values, and others as high as 472.71%, indicating that they are up to five times more efficient than the baseline estimator.

The results in Table 4, biases range from approximately from −0.3241 to 1.0656 for the existing estimators. The existing estimator recorded biases of −0.4135 and −0.0366, and another, 9.5990 and 0.2586 under the Truth and False values respectively. However, the proposed neutrosophic estimators demonstrate smaller bias values overall. Several of them record minimal biases, generally below 0.6 indicating that the proposed estimators provide estimates that are practically unbiased. These results confirm that the modifications introduced in the proposed estimators enhance their stability and reduce systematic error even under large sample condition. The Mean Square Error (MSE) values also show distinct differences between the existing and proposed estimators. For existing estimators, MSE values under Truth value and False value vary widely, ranging from 0.4799 to 76.2431 (for Truth values) and from 0,4607 to 7078.983 (False values). On the other hand, the proposed estimators exhibit markedly smaller MSEs. Most members of the proposed estimators record MSE values below 9.3 under both the Truth and False values, such as T_{P22j} and T_{P34j}. This reduction in MSE indicates that the proposed estimators are more precise and stable. The Percent Relative Efficiency (PRE) results provide further additional evidence of the improvement achieved by the proposed estimators. Among existing estimators, PRE values vary considerably from 18.847% and 867.859% under Truth values and from 21.31% to 527.059% under F-values implying moderate levels of efficiency relative to the baseline estimators.

From table 5, it is observed that the proposed estimators exhibit smaller and more consistent bias values, ranging from 0.0027 to 1.8819, indicating estimates very close to the true population median. On the other hand, traditional and existing estimators show higher and more variable biases, ranging from −0.5576 to 1.5897, reflecting reduced stability under uncertainty. The MSE results reinforce these findings. Traditional estimators display relatively large MSEs, some exceeding 5, signaling lower precision and high variability. Conversely, the proposed Neutrosophic estimators achieve substantially lower MSEs, between 1.0380 and 5.5721, demonstrating improved accuracy and robustness even under indeterminate conditions. PRE values further highlight the advantage of the proposed approach. While traditional estimators record efficiencies from 11.406% to 964.857%, the proposed estimators achieve much higher values, ranging from 18.043% to 578.070% under Truth values and 55.506% to 578.070% under False values. Some estimators, reaching 315.254% and 360.004%, perform several times better than the baseline, indicating a marked gain in efficiency.

Overall, the proposed Regression-cum-Exponential Type Neutrosophic Estimators consistently outperform existing methods, offering lower bias, reduced MSE, and higher PRE. he enhanced performance is attributable to the integration of regression adjustment, exponential transformation and neutrosophic logic, which collectively improve reliability and precision in the presence of uncertainty. These findings establish the proposed estimators as a robust and efficient alternative for population median estimation under indeterminate conditions.

CONCLUSION

This paper considered developing of neutrosophic estimators of population median using auxiliary variables. The theoretical properties (Biases and MSEs) of the proposed estimators up to first order approximation were derived and presented. Empirical studies to assess the performance the proposed estimators were conducted through simulation process. From the results obtained from the empirical study on the efficiency of the proposed estimators relative to the existing estimators examined in this research, it is evident that the proposed estimators consistently exhibit minimum Mean Squared Error (MSE) across all numerical computations. This demonstrates that the proposed estimators possess a higher level of efficiency compared to the other estimators considered in the study. The numerical analysis further confirms that all five proposed estimators provide more accurate and reliable estimates**,** highlighting their superiority and practical relevance in estimating the population median under the Neutrosophic framework.

REFERENCE

Abu-Dayeh, W. A., Ahmed, M. S., Ahmed, R. A., and Muttlak, H. A. (2003). Some Estimators of a Finite Population Mean Using Auxiliary Information. *Applied Mathematics and Computation,* **13**(9), 287-298.

Arcos, A., Rueda, M., and Martinez-Miranda, M. D. (2005). Using multiparametric auxiliary information at the estimation stage. *Statistical Paper,* **4**(3), 339-358.

Audu, A., Ishaq, O. O., R. V. K. Singh, A. Danbaba and Manu, F. (2023). On the study of efficiency of exponential-type estimator of population mean using robust regression methods. *Quality and Reliability Engineering International*, 39,190-205.

Audu, A., Ishaq, O. O, Muili, J. O., Abubakar, A. Rashida, A, Akintola, K. A. Isah, U (2020). Modified Estimators of Population Mean Using Robust Multiple Regression Methods. NIPES Journal of Science and Technology Research 2(4), 12 – 20.

Bahl, S., and Tuteja, R. K. (1991). Ratio and product type exponential estimators. *Journal of Information and Optimization Sciences*, **12**(1), 159-163.

Cochran, W. G. (1940). The estimation of the yields of cereal experiments by sampling for the ratio of grain to total produce. *The journal of agricultural science*, **30**(2), 262-275.

Garci`a, M. R., and Cebrian, A. A. (2001). On estimating the median from survey data using multiple auxiliary information. *Metrika,* **54**(1), 59-76.

Gross, S. (1980). Median estimation in sample surveys. *In Proceeding of the section on survey research methods, American Statistical Association Alexandria,* VA.

Hussain, M. A., Javed, M., Zohaib, M., Shongwe, S. C., Awais, M., Zaagan, A. A., and Irfan, M. (2024). Estimation of population median using bivariate auxiliary information in simple random sampling. *Heliyon*, **10**(7), 1-14.

Irfan, M., Javed, M., Shongwe, S. C., Zohaib, M., and Haider Bhatti, S. (2021). Estimation of population median under robust measures of an auxiliary variable. *Mathematical Problems in Engineering***,** 1-14.

Kuk, A.Y.C., and Mak, T.K. (1989). Median Estimation in the Presence of Auxiliary Information. *Journal of the Royal Statistical Society* **51**(2), 261-269.

Lamichhane, R., Singh, S. and Diawara, N. (2017). Improved estimation of population mean using known median of auxiliary variable. *Commun. Statistical Simulation Computation*. **46**, 2821-2828.

Masood, S., Ibrar, B., Shabbir, J., Shokri, A., and Movaheedi, Z. (2024). Estimating neutrosophic finite median employing robust measures of the auxiliary variable. *Scientific Reports*, **14**(1), 16504.

Mishra, P., Adichwal, N. K., and Singh, R. (2017). A new log-product type estimator using auxiliary information. *Journal of Science Research*, **61**(1), 179-183.

Muneer, S., Khalil, A., Shabbir, J., and Narjis, G. (2020). Efficient estimation of population median using supplementary variable. *Scientific Irancia,* doi:10.24200/2020.52871.2924.

Murthy, M. N. (1964). Product method of estimation. *Sankhyā: The Indian journal of statistics, series A,* **26**(1), 69-74.

Rao, J. N. K., kovar, J. G., and Mantel, H. J. (1990). On estimating distribution functions and quantiles from survey data using auxiliary information. *Biometrik*a, **77**(2), 365-375.

Sharma, P., and Singh, R. (2014). Efficient class of estimators for population median using auxiliary information. doi: 10.48550/ arXiv.1402.5609.

Sharma, P. S. P., and Singh, R. (2015). Generalized class of estimators for population median using auxiliary information. *Hacettepe Journal of Mathematics and Statistics*, **44**(2), 443-453.

Sher, K., Iqbal, M., Ali, H., Iftikhar, S., Aphane, M., Audu, A. (2025). Novel efficient estimators of finite population mean in simple random sampling. *Scientific African*, 27, 1- 12.

Singh, R. V. K. and Audu, A. (2015**)**: Improved Exponential Ratio–Product Type Estimator for finite Population Mean. *International Journal of Engineering Science and Innovative Technology*, 4(3), 317-322

Singh, R., Mishra**,** P., Audu, A. and Khare, S. (2020): Exponential Type Estimator for Estimating Finite Population Mean. *Int. J. Comp. Theo. Stat*., 7(1), 37-41

Singh, R., and Singh, H. P. (1998). Almost unbiased ratio and product type estimators in systematic sampling. *Questiio*, **22**(3), 403-416.

Singh, S., Singh, H. P., and Upadhyaya, L.N. (2007). Chain ratio and regression type estimators for median estimation in survey sampling. *Statistical paper*, **48**(1), 23-46.

Singh, H. P., and Solanki, R. S. (2013). Some Classes of Estimators for the Population Median Using Auxiliary Information. *Communication Statistical Theory Methods*, **42**(23), 4222- 4238.

Smarandache, F. (1998*)*. *Neutrosophy: Neutrosophic Probability, set, and Logic*. *Rehoboth:* American Research Press.

Smarandache, F. (2014). *Introduction to neutrosophic Statistics*. Craiova, Romania: Sitech and Education Publising, doi/10.48550/arXiv.1406.2000.

Singh, R., Kumari, A., Smarandache, F., and Tiwari, S. N. (2024). Construction of Almost Unbiased Estimator for Population Mean Using Neutrosophic Information. *Neutrosophic Sets System,* **76**(1), 24.

Singh, R., and Tiwari, S. N. (2025). Improved Estimator for Population Mean Utilizing Known Medians of Two Auxiliary Variables under Neutrosophic Framework. *Neutrosophic Systems with Applications*, **25**(1), 38-52.

Singh, R., Kumari, A., Smarandache, F., and Yadav, S. K. (2025). Construction of Almost Unbiased Estimator for Population Median Using Neutrosphic Information. *Pakistan Journal of Statistics***, 41**(2), 203-219.

Tahir, Z., Khan, H., Aslam, M., Shabbir, J., Mahmood, Y., and Smarandache, F. (2021). Neutrosophic ratio-type estimators for estimating the population mean. *Complex and Intelligent Systems*, **7**(6), 2991-3001.

Vishwakarma, G. K., and Singh, A. (2023). Generalized estimator for computation of population mean under neutrosophic ranked set technique: Application to solar energy data. *Computational and Applied Mathematics*, **41**(4), Article 144. doi/10.1007/s40314-022- 01820-7.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, **8**(3), 338-353. doi/10.1016/S0019-9958(65)90241-X.