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Signature reliability of linear consecutive k-out-of-n:G system using universal generating function approach

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ABSTRACT

This paper investigates the signature reliability of a linear consecutive k-**out-of-n:**G system using the universal generating function (UGF) technique. The system comprises n components, where only k consecutive ones need to function for the entire system to operate. As a starting point, a linear consecutive 4-out-of-7: G system is modeled to derive its reliability function, which is then converted into polynomial form. The study next analyzes the system's tail-signature to support the evaluation of its signature reliability. facilitate the evaluation of its signature reliability. The Barlow-Proschan index is applied to assess the system's overall reliability. Finally, the study derives key performance metrics including the minimal signature, expected lifespan, and actual cost rate for the proposed configuration.

INTRODUCTION

A linear consecutive k-out-of-n system includes n components arranged sequentially. In such a system, a k-**out-of**-n:G system operate successfully if at least k consecutive components are functioning. On the other hand, a k-**out-of**-n:F system is considered to have failed if at least k consecutive components are not working **Levitin, (2005).** These two systems are duals of each other Kuo, (2012). This implies that when the reliability of each component in a k-**out-of**-n:G system is equals to the unreliability of the corresponding component in a k-**out-of**-n:F system, the total reliability of the k-**out-of**-n:G system equals the total unreliability of the k-**out-of**-n:F system. Consequently, the same reliability evaluation methods apply to both. This study focuses solely on k-**out-of**-n:G systems.

The concept of linear consecutive k-**out-of**-n systems was formally introduced by Chiang and Niu, (1981), though Kontoleon, (1980) had earlier alluded to it. Early reliability evaluation methods for systems with identical components were proposed by Chiang, (1986), Bollinger, (1982), Derman, (1972) and Hwang, (1986). Later, Zuo, (1990) addressed more complex cases involving non-identical components. These systems are widely used in engineering, including:

Relay stations, Chiang, (1986), Street parking systems, Kuo et al. (1990), Oil pipelines, Chang (1998) Quality control systems Kuo, (2012), Street lighting Yuan, (2013). Components in these systems must be functionally identical, though their reliabilities may differ Bennour, (2015). A key feature is configurability reliability can be improved by rearranging components Wang, (2020).

Further enhancements can be achieved through Increasing component reliability, adding redundancy and optimizing component placement. Juan Yin and Lirong Cui, (2021) extended the model to include shared components between subsystems, covering linear, circular, zigzag, and polygonal configurations. A major focus in this field is cost-effective reliability improvement, combining multiple strategies such as component maintenance, rearrangement and redundancy. The field of signature reliability remains an active area of research.

This study builds upon prior work by Levitin, (2005), who evaluated the reliability of a linear consecutive 3-out-of-5 system using the Universal Generating Function (UGF) approach. Additionally, Sadiya et al., (2024) analyzed a circular consecutive 3-out-of-5: W system, comparing structure function and UGF methods for reliability assessment.

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In this research, we apply the UGF approach to evaluate the reliability function and signature reliability of a linear consecutive 4-out-of-7: G system. Universal Generating Function (UGF) Approach, introduced by Levitin and Lisnianski, (1999) is a mathematical technique for assessing the reliability of complex systems. It represents system components and their reliability properties through a polynomial function, enabling the calculation of system reliability and performance metrics Levitin, (2005). Bentolhoda Jafar and Lance Fiondella, (2015) extended the discrete UGF approach to multi-state systems, incorporating component correlations and generalizing failure distributions. Kumar, (2017) applied UGF to evaluate the reliability of bridge structures, demonstrating its effectiveness in real-world engineering systems. Tyagi S. et al., (2020) used UGF to assess renewable energy systems, estimating key metrics such as system signature and minimal signature, tail signature, expected lifetime and cost rate, Barlow-Proschan index. Goyal et al., (2012) studied series-parallel systems with independent and identically distributed (i.i.d.) components, deriving reliability measures including mean time to failure (MTTF). Sadiya et al., (2024) compared structure function and UGF approaches for a consecutive 3-out-of-5: W system. Their findings indicated that the structure function method is preferable due to its simplicity, despite both methods yielding consistent results.

Over the past four decades, signature reliability has been a prominent area of research in complex systems. The foundation work by Samaniego, (1985) introduced the concept of system signatures for coherent systems with independent and identically distributed (i.i.d.) lifespans. Signature reliability is a data-driven approach that identifies system failure patterns to predict its reliability. Notable advancements in this area include Marichal and Mathonet, (2013) demonstrated efficient signature computation using derivatives of the reliability function's diagonal section. Marichal, (2014) derived a general formula for system signatures based on partitioned modules. Gaofeng et al. developed an algorithm for computing signatures in systems with exchangeable components, relying solely on minimal cut or path sets. Emad, (2015) applied path-tracing methods to evaluate communication system reliability, facilitating probabilistic signature analysis.

MATERIALS AND METHODS

The signature reliability of the system is evaluated by the following algorithms.

Algorithm for reliability function evaluation using UGF approach.

The following recursive procedure determines the reliability of a linear consecutive k-out-of-n system with non-identical components Levitin, (2005).

1.Determine u-function for element in the form of eq.(1)

2. Initialize $U_1(Z) = u_1(z)$

3. For $j = 2 \cdots n$ compute $U_j(z) = U_{j-1}(z) \otimes u_j(z)$, (the final u-function $U_n(z)$represents the probability mass function of random variable X)

4. Obtain the system's structure function via $U(z) = U_n(Z) \otimes \varphi k$, *where* $\varphi(X, K) = 1(X \geq k)$

5. Compute system's reliability as $E(\varphi\ (X,k) = U'(1) = \text{R}$.

For the proposed linear consecutive **4-out-of-7: G system** component reliabilities $E_1, E_2, \cdots E_7$. are used to derive the general u-function of the system**.** The UGF for each component in the system is given by:

$$U_a(z) = x_a z^1 + (1 - x_a) z^0, \tag{1}$$

where a = 1,2, …,7 are component index and X_a is the likelihood function (reliability of component a) $\mathbf{z^a}$ is performance state**,** and $\mathbf{z}^0$ is failure state.

Algorithm for system signature computation

The system signature is evaluated using the Boland formula**,** (2001).

$$C_a = \frac{1}{\binom{n}{n-a+1}} \sum^{n}_{\substack{k\subseteq[n] \\ |k|=n-a+1}} \phi(k) - \frac{1}{\binom{n}{n-1}} \sum^{n}_{\substack{k\subseteq[n] \\ |k|=n-1}} \phi(k) \tag{2}$$

The reliability polynomial:

$$\mathbf{K(P)} = \sum_{J=1}^{n} e_i \binom{n}{j} p^j q^{n-j} \tag{3}$$

Where $\mathbf{e_i} = \sum_{i-n-j+1}^{n} b_i$ **j = 1, 2, …n** (4)

This calculation incorporates the system's reliability function. The resulting polynomial form, derived from this method, is then used to determine the device's reliability function by applying a Taylor series expansion evaluated at $\omega = 1$. The polynomial function's formula is expressed as: Kumar (1)

$$\text{R}(\omega) = \omega^n K(\tfrac{1}{\omega}) \tag{5}$$

The tail signature, denoted as a (p + 1)-tuple $B = (B_1, B_2, \dots, B_n)$, need to be estimated. It is computed using the following formula:

$$\text{B}_a = \sum_{i=a+1}^{n} b_i = \frac{1}{\binom{n}{n-1}} \sum_{|H|=n-1}^{n} \phi(k) \tag{6}$$

By Marichal and Mathonet (2013) and Eq. (6) the tail signature of the system is calculated as follow:

$$B_a = \frac{(n-1)!}{n!} d^a p(1), \quad a = 1, 2, \dots, n \tag{7}$$

From this, the system's signature is obtained as:

$$B = \text{B}_a - \text{B}_{a-1} \qquad a = 1,2,\cdots n \tag{8}$$

Algorithm for expected lifetime of the system

The expected lifespan of the system is determined using the minimal signature. It is assumed that the system's components in the system are i.i.d. According to Navarro and Rubio, (2019) the expected lifetime of i.i.d. components can be estimated using the following expression

$$E(T) = \mu \sum_{i=1}^{n} \frac{e_i}{i} \quad (9)$$

where $e = (e_1, e_2, \ldots, e_n)$ is the minimal signature vector.

Algorithm for B*arlow-P*r*oschan* I*ndex*

For i.i.d. components, the Barlow-Proschan index (BPI) measures component importance. By using Shapley, (1953) and Owen, (1972) formula the Barlow-Proschan index is determined as follows:

$$BPI^a = \int_0^n (\delta_a R)(w)dw, \quad a = 1,2,\cdots,n. \quad (10)$$

where R is the reliability function of the system.

Algorithm for expected system value and cost rate

The expected value of the system is calculated as follows:

$$E(X) = \sum_{i=1}^{n} iB_i \quad , i = 1,2,\cdots n \quad (11)$$

The expected cost rate is derived as:

$$\frac{E(X)}{E(T)} \quad (12)$$

N*umerical illustration*

Considering a linear consecutive 4-out-of-7:G system, The system consisted of four working components and three fail components. In which the components are connected in a linear manner.

RESULTS AND DISCUSSION

By using the algorithms (1), the individual component UGFs are: Sadiya et al, (2024)

$$u_1(z) = x_1 z^1 + (1 - x_1)z^o$$
$$u_2(z) = x_2 z^1 + (1 - x_2)z^o$$
$$\vdots$$
$$u_n(z) = x_n z^1 + (1 - x_n)z^o$$

$$U(Z) = \prod_{i=1}^{7} x_i + (1\text{-}x_1)\prod_{i=2}^{7} x_i + x_1(1\text{-}x_2)\prod_{i=3}^{7} x_i + \prod_{i=1}^{2} x_i(1\text{-}x_3)\prod_{i=4}^{7} x_i + \prod_{i=1}^{4} x_i(1\text{-}x_5)\prod_{i=6}^{7} x_i + \prod_{i=1}^{5} x_i(1\text{-}x_6)x_i + \prod_{i=1}^{6} x_i(1\text{-}x_7) + \prod_{i=1}^{2}(1-x_i)\prod_{i=3}^{7} x_i + \prod_{i=1}^{3}(1-x_i)\prod_{i=4}^{7} x_i + (1\text{-}x_1)\prod_{i=2}^{5} x_i(1\text{-}x_6)x_7 + (1\text{-}x_1)\prod_{i=2}^{6} x_i(1\text{-}x_7) + x_1\prod_{i=2}^{3}(1-x_i)\prod_{i=4}^{7} x_i + x_1(1-x_2)\prod_{i=2}^{6} x_i(1-x_7) + \prod_{i=1}^{3}(1-x_i)\prod_{i=4}^{7} x_i + \prod_{i=1}^{4} x_i\prod_{i=5}^{6}(1-x_i)x_7 + \prod_{i=1}^{4} x_i(1-x_5)x_6(1-x_7) + \prod_{i=1}^{4} x_i\prod_{i=5}^{7}(1-x_i) \quad (13)$$

Therefore,

$$U(Z) = x_1x_2x_3x_4 + x_4x_5x_6x_7 - x_1x_2x_3x_4x_6 + x_2x_3x_4x_5x_7 + x_1x_3x_4x_5x_6 - 2x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_6x_7 - x_1x_2x_3x_4x_5x_7 - x_1x_3x_4x_5x_6x_7 - 2x_2x_3x_4x_5x_6x_7 + 2x_1x_2x_3x_4x_5x_6x_7 \quad (14)$$

Therefore, the reliability function of the system, assuming all components are identically and independently distributed (i.i.d.), becomes:

when

$x_1 = x_2 \cdots, x_7 = x$

Then the reliability function is:

$$R = 2x^4 + 2x^5 - 5x^6 + 2x^7 \quad (15)$$

Signature of the Considered Structure

To evaluate system signature first transform reliability function into polynomial function system using eq. (5):

$$P(R) = R^n K(\tfrac{1}{R})$$
$$= x^7(\tfrac{2}{x^4} + \tfrac{2}{x^5} - \tfrac{5}{x^6} + \tfrac{2}{x^7})$$
$$= 2 - 5x + 2x^2 + 2x^3 \quad (16)$$

By using equation (7) the tail signature B of the system is as follow:

$$B_a = \frac{(n-1)!}{n!} d^a p(1), \quad a = 0, 1, \ldots, n$$

$$B = (1, \tfrac{5}{7}, \tfrac{8}{21}, \tfrac{6}{105}, 0, 0, 0) \quad (17)$$

The tail signature ***B*** is the probability that the system survives beyond the k-th component failure. Base on this statement, initially the system is in perfect state, at the first failure, system survive with approximate 71.4% at second 38.1%, third failure 5.7%, system does not survive beyond four plus failures.

Consequently, using tail signature values, the system's signature is determined by using eq.(8)

$$S = B_a - B_{a-1}$$

where *a* = 1,2,⋯7

$$S = (\tfrac{2}{7}, \tfrac{1}{3}, \tfrac{34}{105}, \tfrac{6}{105}, 0, 0, 0) \quad (18)$$

System's signature is the probability vector S that described the likelihood of the system failing at the K-th component failure. This means: there is approximate 28.57% chance the system fails at the first failure, 33.33% chance it fails at the second, 32.38% chance at third failure, 5.71% at the fourth failure, 0% for fifth and beyond. system always fails earlier due to the consecutive requirement.

B*arlow* P*roschain index of the considered system*

With the help of equation (10) the Barlow proschain index of the system as follow:

$$BPI^a = \int_0^n (\delta aR)(w)dw, \quad a = \mathbf{1,2,}\cdots\mathbf{,n.}$$

$$BPI_1 = \int_0^1 (x^3 - 3x^5 + 2x^6)dx = \frac{5}{84} \quad (19)$$

Similarly, Barlow-Proschan index for ***k* = (**1,⋯ ,7**)** of all elements are as follow:

$\text{BPI}_K = (\frac{5}{84}, \frac{29}{420}, \frac{43}{420}, \frac{37}{105}, \frac{19}{140}, \frac{33}{140}, \frac{33}{140})$ (20)

The Barlow-Proschan index quantifies the importance of a component by its contribution to system failure. Barlow & Proschan, (1975)
This vectors represent the importance of each of the seven components in a 4-out-of-7:G system. Each fraction corresponds to how critical that component is to the overall system's functioning, components with higher BPI values are more critical to system performance. For instance, component 7(with $\frac{33}{140}$) has the highest importance. This values can help in prioritizing maintenance, redundancy, or component upgrades.

E*xpected* L*ifetime of the* S*ystem*

Expected lifetime (lifespan) is the average time of the system before the first failure. Which was evaluated using Eq. (15), from above we get the minimal signature M of the linear consecutive 4-out-7:G system as:

Minimal signature = (0,0,0,2,2, -5,2) (22)

Using minimal signature, we obtain expected lifetime of the system as:
$E(T) = \mu \sum_{i=1}^{n} \frac{e_i}{i} = 0.35$ (23)

E*xpected* C*ost* R*ate*

The expected cost rate integrates economic and reliability metrics to optimize maintenance strategies. Navarro et al, (2007) using algorithm (6), the expected value of the system is determined as:
$E(X) = \sum_{i=1}^{n} iB_i$, i = 1,2,⋯n
$= S_1 + 2S_2 + 3S_3 + 4S_4 + 5S_5 + 6S_6 + 7S_7 = 2.15$ (24)

Expected cost rate = $\frac{E(X)}{E(T)}$ = 6.14 (25)

CONCLUSION

The reliability function with the help of UGF for a linear consecutive 4-out-of-7:G system having equal reliability with i.i.d. elements has been computed. The end concludes, some fundamental results are tail signature (1, $\frac{5}{7}$, $\frac{8}{21}$, $\frac{6}{105}$, 0, 0, 0), System signature ($\frac{2}{7}$, $\frac{1}{3}$, $\frac{34}{105}$, $\frac{6}{105}$, 0, 0, 0), Barlow-Proschan index ($\frac{5}{84}, \frac{29}{420}, \frac{43}{420}, \frac{37}{105}, \frac{19}{140}, \frac{33}{140}, \frac{33}{140}$), Minimal signature=(0,0,0,2,2,-5,2), expected lifetime = 0.35, expected cost rate = 6.14 using Owen's method, have been determined.
This work allows one to expand in this area by adding more redundant components example linear consecutive of 4 and number above 7, high bridge system as well as numerous other systems.

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