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A Discrete Time Economic Order Quantity Model for Ameliorating Items with Constant Holding Cost and Linear Demand

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ABSTRACT
This research is concerned with the creation of an inventory model for improving items with discrete time, constant holding cost, and linear demand. When products are successfully in stock, they undergo some improvement, and as a result, the stock becomes more useful.
The aim is to find the optimal order size and restocking interval that keep total costs as low as possible. The study presents a framework that analyses how the increasing value of items affects inventory decisions and optimal ordering quantities. The article went further to discuss how linear demand cost influence inventory levels and discuss optimization techniques that can be employed to efficiently manage these costs. The demand rate is considered to be linear, and shortages are not permitted. The model is numerically illustrated, and a sensitivity analysis with regard to the ameliorating factors is performed to determine the effect of parameter modifications.

Keywords:
Amelioration,
Discrete time,
Economic order quantity (EOQ),
Constant holding cost and linear demand.

INTRODUCTION

The EOQ model is one of the most widely utilized in inventory management. Ford W. Harris proposed it in 1913, and many businesses have utilized it since then. Numerous research initiatives have been made to expand the study of models to include real-life scenarios. In inventory models, researchers initially assumed a constant demand rate; however, Ghare and Schrader (1963) were among the first to analyse inventory systems for items with linear demand and a constant rate of deterioration. Hwang (1997) was the first to provide models for improving goods. His article focused on commodities that improve over time when in stock.

The approach sought to identify the ideal settings that would cut costs while increasing profits.

Hwang (1999) extended the economic order quantity framework by adding partial selling-price effects and inventory issuing rules such as first-in-first-out (FIFO) and last-in-first-out (LIFO). He modelled product improvement and deterioration with a two-parameter Weibull distribution and incorporated factors like amelioration, selling price, and demand. The model explicitly accounts for degradation that occurs while items are in stock, using the Weibull distribution to represent both improvement and decay.

Biswajit et al. (2003) later examined product improvement underprice-dependent demand within an immediate-replenishment system where shortages are not permitted.

Barman et al. (2010) created an economic order quantity model with amelioration. The model accounts for the cost of amelioration, the scarcity cost, and the backordering cost. Gaur et al. (2014) developed an EOQ model with amelioration and a backlog policy. The model envisions a production system in which defective things might be improved and then sold at a discount. Han-Wen et al. (2017) created a model for improving items with Weibull distributions and determining the best solution to the problem. Chao and Hsu (2016) developed an economic order quantity model with amelioration that takes into account a system where the amelioration cost is linear.

Gwanda (2018) proposed an economic order quantity model to improve products with time-dependent demand and linear time-dependent holding costs. Chen et al. (2018) created an EOQ model that includes amelioration in an imperfect production system. Shakya and Shah (2018) investigated an EOQ model with improved faulty items and a linear pricing system. It considers an EOQ model in which both demand and price have a linear relationship.

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Yahaya et al. (2019) developed an EOQ model introducing policy to improve inventory with a linear demand rate and unconstrained retailer capital. Barman et al. (2019) investigated a stochastic EOQ model with amelioration. The paper examined a system in which demand is unpredictable and amelioration time is a random variable.

Razzaque et al. (2019) investigated an EOQ model with improved defective items under uncertain demand and procurement costs. It examined the EOQ approach for improving under unknown demand and procurement costs. Demand is unknown; therefore, inventory costs are also probabilistic. Mohammed and Ahmed (2019) proposed a stochastic EOQ model that includes amelioration and pricing negotiation. The paper used a new EOQ model with amelioration and price negotiation to investigate the impact of amelioration and price negotiation on the ordering policy of a single-vendor single buyer system.

The authors considered that demand varies owing to market fluctuations, and that the vendor's selling price varies according to inventory levels. Yazici et al. (2019) created an EOQ model that improves defective products under variable demand and supply from two vendors. Luo et al. (2020) created an EOQ model that addresses defective items through a multi-step delivery strategy with price negotiation. The study looks at where a buyer can negotiate the selling price with the seller and how the vendor offers different rates depending on the delivery schedule.

Products with a discrete time demand pattern have intermittent and variable demand, with the number of orders placed each time being the same as the product's net needs for that time. Time is measured in terms of full days, weeks, months, or years. There are researchersthat have work on discrete-time inventory models. These includes Aliyu and Boukas (1998), that talked about instant replacement for the items over a set length of time with deterministic demand and discrete time .Zhaotong and Liming (1999) created a discrete time model for perishable inventory systems that includes geometric inter-demand times and batch demands.

Ferhan et al. (2013) investigated an inventory model for deteriorating productsformulated in a non-periodic discrete time structure, inwhich time points are not uniformly distributed. Yakubu and Sani (2015) suggested an EOQ model for degrading objects with a discrete time and delayed deterioration. Adamu and Yakubu (2024) contributed to this chain of research by developing a discrete time EOQ model for ameliorating items under constant demand. The constant demand case makes it suitable to industries where demand remains stable and predictable. This may not be suitable for products subject to growth or decline in demand, like fashion goods, seasonal product and technology-based items. For this reason, there is a need to close the gap as linear demand may be more realistic, cost-accurate and flexible for products with time -varying demand.

In this research, we investigate a discrete time EOQ model for Amelioration items with constant holding cost and linear demand. Improvement occurs when items are held in stock. The inventory commences with simultaneous demand and amelioration, continuing until the stock reaches zero at t = T.

This work seeks to establish the optimal ordering policy specifically, the order size and cycle length that leads to cost minimization. A numerical example is solved to demonstrate the model's applicability. Finally, a sensitivity analysis is performed to assess the impact of parameter changes on the decision factors.

NOTATION AND ASSUMPTIONS

Notation

$I_A(t)$ Inventory level at time t

C_0 The ordering cost

$I_A(0)$order quantity

T Cycle length

A_m Amount due to Amelioration

C_H Cost of holding stock per replenishment cycle

D_TOverall demand in a cycle of length, T.

iUnit carrying charge

C Item cost per unit

D Demand rate($D = \alpha_1 + \alpha_2 t$) (unit / time)

TVC Total variable cost per replenishment cycle

A Constant rate of Amelioration

Assumptions

(i) Zero lead time is assumed

(ii) A single item undergoing instantaneous amelioration is considered.

(iii) Shortages are not allowed

(iv) Amelioration occurs immediately the items arrive in stock

MATERIALS AND METHODS

Model Formulation

A typical behaviour of the inventory in a cycle is depicted in figure 1 below. From the beginning, the inventory cycle commences with demand and amelioration occurring simultaneously, and the stock reduces progressively until it is fully depleted at t = T. $I_A(0)$ denotes the ordered quantity at the start of the cycle.

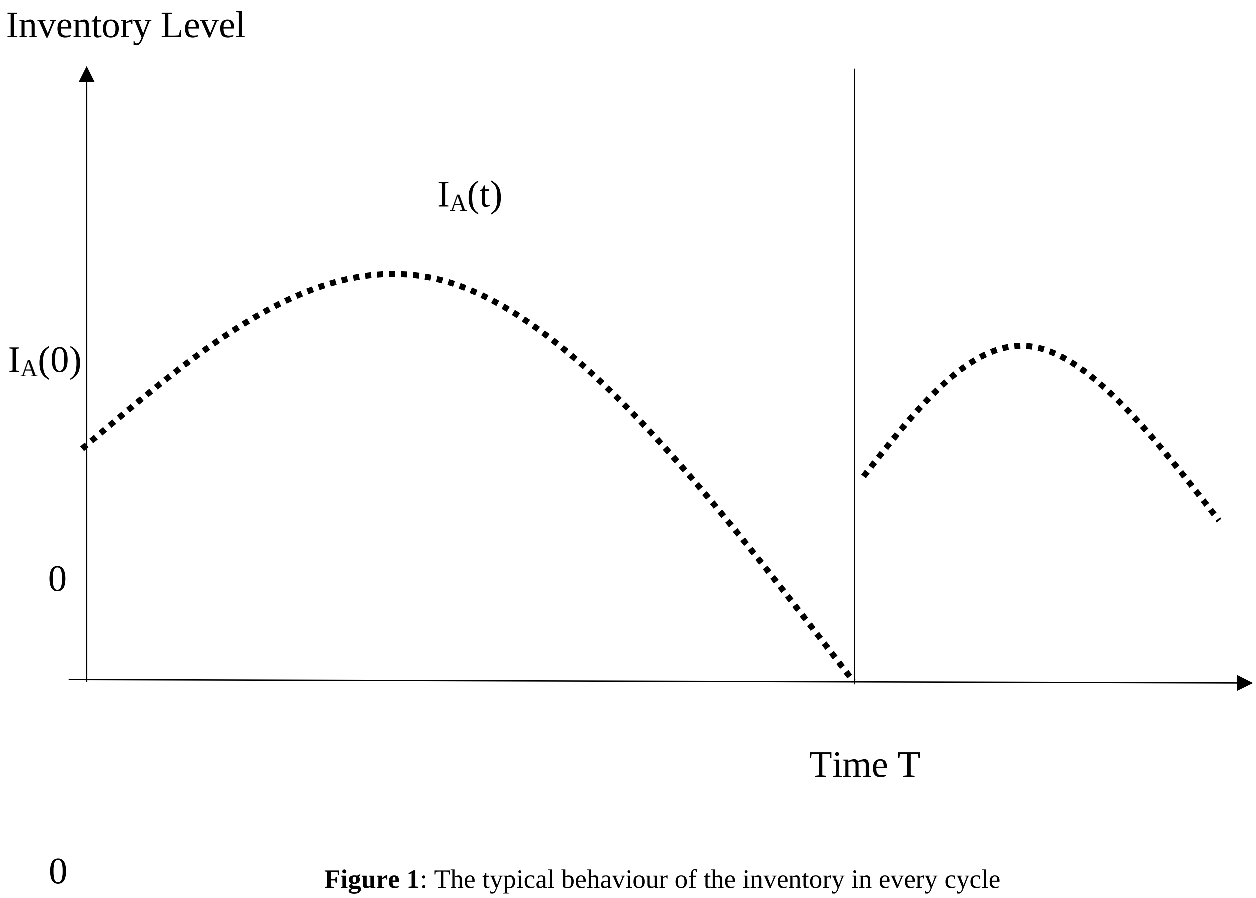

Figure 1: The typical behaviour of the inventory in every cycle

$I_A(0)$ denotes the ordered quantity at the start of the cycle, $I_A(t)$ is the inventory position at time t with interval $(0 \le t \le T)$,Aas a constant rate of amelioration with demand rate $D = \alpha_1 + \alpha_2 t$as a linear function of time.
The evolution of$I_A(t)$is described through the following difference equation

$$\Delta I_A(t) = AI_A(t) - (\alpha_1 + \alpha_2 t)\ 0 \le t \le T \qquad (1)$$

Subject to the initial and boundary conditions $I_A(t) = I_A(0)$, $att = 0$ and $I_A(T) = 0$, $at\ t = T$.
The solution is obtained as follows:
Given that $\Delta f(x) = f(x+h) - f(x)$, with h as the step length, it follows that
$\Delta I_A(t) = I_A(t+1) - I_A(t)$ with step length of 1.
$\Rightarrow I_A(t+1) - I_A(t) = AI_A(t) - (\alpha_1 + \alpha_2 t)$ from (1)
$\Rightarrow I_A(t+1) = (1+A)I_A(t) - (\alpha_1 + \alpha_2 t)$ for t = 0, 1, 2, 3, …, (T-1), T.
For $t = 0$

$$I_A(1) = (1+A)I_A(0) - (\alpha_1 + \alpha_2 0)$$
$$= (1+A)I_A(0) - \alpha_1$$

For $t = 1$

$$I_A(2) = (1+A)I_A(1) - (\alpha_1 + \alpha_2 1)$$
$$= (1+A)I_A(1) - (\alpha_1 + \alpha_2)$$

But $I_A(1) = (1+A)I_A(0) - \alpha_1$

$$\Rightarrow I_A(2) = (1+A)[(1+A)I_A(0) - \alpha_1] - (\alpha_1 + \alpha_2)$$
$$= (1+A)^2 I_A(0) - (1+A)\alpha_1 - (\alpha_1 + \alpha_2)$$

For $t = 2$

$$I_A(3) = (1+A)I_A(2) - (\alpha_1 + \alpha_2 2)$$
$$= (1+A)[(1+A)^2 I_A(0) - (1+A)\alpha_1 - (\alpha_1 + \alpha_2)] - (\alpha_1 + 2\alpha_2)$$
$$\Rightarrow I_A(3) = (1+A)^3 I_A(0) - (1+A)^2\alpha_1 - (1+A)(\alpha_1 + \alpha_2) - (\alpha_1 + 2\alpha_2)$$

For $t = 3$

$$I_A(4) = (1+A)I_A(3) - (\alpha_1 + \alpha_2 3)$$
$$= (1+A)[(1+A)^3 I_A(0) - (1+A)^2\alpha_1 - (1+A)(\alpha_1 + \alpha_2) - (\alpha_1 + 2\alpha_2)] - (\alpha_1 + \alpha_2 3)$$
$$\Rightarrow I_A(4) = (1+A)^4 I_A(0) - (1+A)^3\alpha_1 - (1+A)^2(\alpha_1 + \alpha_2) - (1+A)(\alpha_1 + 2\alpha_2) - (\alpha_1 + 3\alpha_2)$$

Up to t = (T-1),
we have

$$I_A(t) = (1+A)^t I_A(0) - (1+A)^{t-1}\alpha_1 - (1+A)^{t-2}(\alpha_1 + \alpha_2) - (1+A)^{t-3}(\alpha_1 + 2\alpha_2) - (1+A)^{t-4}(\alpha_1 + 3\alpha_2) - (1+A)^{t-5}(\alpha_1 + 4\alpha_2) - (1+A)^{t-6}(\alpha_1 + 5\alpha_2) - \cdots - (1+A)^{t-t}(\alpha_1 + t\alpha_2)$$

So that for t = T, we obtain

$$I_A(T) = (1+A)^T I_A(0) - (1+A)^{T-1}\alpha_1 - (1+A)^{T-2}(\alpha_1 + \alpha_2) - (1+A)^{T-3}(\alpha_1 + 2\alpha_2) - (1+A)^{T-4}(\alpha_1 + 3\alpha_2) - (1+A)^{T-5}(\alpha_1 + 4\alpha_2) - (1+A)^{T-6}(\alpha_1 + 5\alpha_2) - \cdots - (1+A)^{T-T}(\alpha_1 + T\alpha_2)$$

In general, we get

$$I_A(n+1) = (1+A)^{n+1} I_A(0) - \sum_{i-0}^{n} (1+A)^{n-i}[\alpha_1 + i\alpha_2]$$

$$\Rightarrow \; I_A(t) = (1+A)^t I_A(0) - \sum_{i-0}^{t-1} (1+A)^{t-1-i}[\alpha_1 + i\alpha_2] \qquad (2)$$

Going up to T, yields

$$I_A(T) = (1+A)^T I_A(0) - \sum_{i-0}^{T-1} (1+A)^{T-1-i}[\alpha_1 + i\alpha_2] \qquad (3)$$

Using the boundary condition at $t = T$, $I_A(T) = 0$ we have, from equation (3)

$$0 = (1+A)^T I_A(0) - \sum_{i-0}^{T-1} (1+A)^{T-1-i}[\alpha_1 + i\alpha_2]$$

Which simplifies to

$$I_A(0) = \frac{\sum_{i-0}^{T-1} (1+A)^{T-1-i}[\alpha_1 + i\alpha_2]}{(1+A)^T} \qquad (4)$$

Substituting equation (4) in equation (2) we have

$$I_A(t) = (1+A)^t \left[\frac{\sum_{i-0}^{T-1} (1+A)^{T-1-i}[\alpha_1 + i\alpha_2]}{(1+A)^T}\right] - \sum_{i-0}^{t-1} (1+A)^{t-1-i}[\alpha_1 + i\alpha_2] \qquad (5)$$

Taking the summation

$$\sum_{i-0}^{T-1} (1+A)^{T-1-i}[\alpha_1 + i\alpha_2]$$

$$= (1+A)^{T-1}\alpha_1 + (1+A)^{T-2}(\alpha_1 + \alpha_2) + (1+A)^{T-3}(\alpha_1 + 2\alpha_2) + (1+A)^{T-4}(\alpha_1 + 3\alpha_2) + \cdots$$

$$= \alpha_1\left[(1+A)^{T-1} + (1+A)^{T-2} + (1+A)^{T-3} + (1+A)^{T-4} + \cdots + (1+A)^{(T-1)-(T-1)} + 1\right] + \alpha_2\left[(1+A)^{T-2} + 2(1+A)^{T-3} + 3(1+A)^{T-4} + 4(1+A)^{T-5} + (T-2)(1+A) + (T-1)\right]$$

Sum of α_1 components is a geometric series, where the first term is $(1+A)^{T-1}$, the common ratio is equal to $(1+A)^{-1}$ and the number of terms is T.

Thus, the sum is

$$= \alpha_1\left[(1+A)^{T-1}\left(\frac{1-(1+A)^{-1(T+1)}}{1-(1+A)^{-1}}\right)\right]$$

$$= \alpha_1\left[\left(\frac{(1+A)^{T-1} - (1+A)^{-2}}{1-(1+A)^{-1}}\right)\right]$$

Similarly, α_2 components form an arithmetico geometrical series

Where a = 1, b= $(1+A)^{T-2}$, $d = 1$, $r = (1+A)^{-1}$ and is calculated as

$$S_n = \frac{1(1+A)^{T-2} - (1+(T-1)1)(1+A)^{T-2}[(1+A)^{-1}]^{T-1}}{1-(1+A)^{-1}} + \frac{1(1+A)^{T-2}(1+A)^{-1}(1-[(1+A)^{-1}])^n}{[1-(1+A)^{-1}])^2}$$

$$= \frac{(1+A)^{T-2} - (1+(T-1))(1+A)^{T-2}(1+A)^{-T+1}}{1-(1+A)^{-1}} + \frac{(1+A)^{T-3}(1-(1+A)^{-n}}{(1-(1+A)^{-1})^2}$$

$$= \frac{(1+A)^{T-2} - (1+T-1)(1+A)^{T-2-T+1}}{1-(1+A)^{-1}} + \frac{(1+A)^{T-3}[1-(1+A)^{-n}]}{(1-(1+A)^{-1})^2}$$

$$= \frac{(1+A)^{T-2} - T(1+A)^{-1}}{(1-(1+A)^{-1})} + \frac{(1+A)^{T-3} - (1+A)^{T-3-n}}{[1-(1+A)^{-1}]^2}$$

$$= \frac{(1-(1+A)^{-1})[(1+A)^{T-2} - T(1+A)^{-1}] + (1+A)^{T-3} - (1+A)^{T-3-n}}{[1-(1+A)^{-1}]^2}$$

$$= \frac{(1+A)^{T-2} - T(1+A)^{-1} + T(1+A)^{-2} - (1+A)^{T-3-n}}{[1-(1+A)^{-1}]^2}$$

$$= \frac{(1+A)^{T-2} - (1+A)^{T-3-(T-1)} - T(1+A)^{-1} + T(1+A)^{-2}}{[1-(1+A)^{-1}]^2}$$

$$= \frac{(1+A)^{T-2} - T(1+A)^{-1} + T(1+A)^{-2} - (1+A)^{-2}}{[1-(1+A)^{-1}]^2}$$

$$S_n = \frac{(1+A)^{T-2} - T(1+A)^{-1} + (T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}$$
$$= \alpha_2\left[\frac{(1+A)^{T-2} - T(1+A)^{-1} + (T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]$$

$$\therefore \sum_{i=0}^{T-1}(1+A)^{T-1-i}[\alpha_1+\alpha_2]$$
$$= \alpha_1\left[\frac{(1+A)^{T-1} - (1+A)^{-2}}{1-(1+A)^{-1}}\right] + \alpha_2\left[\frac{(1+A)^{T-2} - T(1+A)^{-1} + (T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]$$

$$\Rightarrow I_A(t) = \frac{(1+A)^t}{(1+A)^T}\left\{\alpha_1\left[\frac{(1+A)^{T-1} - (1+A)^{-2}}{1-(1+A)^{-1}}\right] + \alpha_2\left[\frac{(1+A)^{T-2} - T(1+A)^{-1} + (T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]\right\} - \sum_{i-0}^{t-1}(1+A)^{t-1-i}[\alpha_1 + i\alpha_2] \quad (6)$$

Also $\sum_{i=0}^{t-1}(1+A)^{t-1-i}(\alpha_1+i\alpha_2) = \alpha_1[(1+A)^{t-1} + (1+A)^{t-2}(\alpha_1+\alpha_2) + (1+A)^{t-3}(\alpha_1+2\alpha_2) + (1+A)^{t-4}(\alpha_1+3\alpha_2) + (1+A)^{t-5}(\alpha_1+\alpha_2) + \cdots + (\alpha_1 + (t-1)\alpha_2)]$ so that for α_1 components, we get

$$\alpha_1[(1+A)^{t-1} + (1+A)^{t-2} + (1+A)^{t-3} + (1+A)^{t-4} + (1+A)^{t-5} + \cdots + (t-1)] + \alpha_2[(1+A)^{t-2} + 2(1+A)^{t-3} + 3(1+A)^{t-4} + 4(1+A)^{t-5} + 5(1+A)^{t-6} + \ldots + (t-1)]$$

This implies the sum of α_1 components will be $\alpha_1\left[\frac{(1+A)^{t-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]$.

In a similar way, $\alpha_2$components form an arithmetico geometrical series where

$$a = 1, \quad d = 1, \quad b = (1+A)^{t-2}, \quad r = (1+A)^{-1}, n = t-1$$

and is simplified as

$$\alpha_2\left[\frac{(1+A)^{t-2} - t(1+A)^{-1} + (t-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]$$

However, substituting these values in equation (5) yields

$$I_A(t) = \left[\frac{(1+A)^t}{(1+A)^T}\left\{\alpha_1\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\right\} + \alpha_2\frac{(1+A)^t}{(1+A)^T}\left\{\frac{(1+A)^{T-2}-T(1+A)^{-1}+(1+A)^{-2}(T-1)}{[1-(1+A)^{-1}]^2}\right\}\right] - \left[\left\{\alpha_1\left[\frac{(1+A)^{t-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right] - \alpha_2\left[\frac{(1+A)^{t-2}-t(1+A)^{-1}+(1+A)^{-2}(t-1)}{[1-(1+A)^{-1}]^2}\right]\right\}\right] \quad (7)$$

Let $(1+A) = P$, then

$$I_A(t) = \frac{P^t}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{t-1}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{t-2}-tP^{-1}+P^{-2}(t-1)}{[1-P^{-1}]^2}\right]\right\}\right]$$

At t = 0, we have

$$I_A(0) = \frac{P^0}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{0-1}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{0-2}-(0)P^{-1}+P^{-2}(0-1)}{[1-P^{-1}]^2}\right]\right\}\right]$$

$$= \frac{1}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{-1}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{-2}-P^{-2}}{[1-P^{-1}]^2}\right]\right\}\right]$$

At $t = 1$

$$I_A(1) = \frac{P^1}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{1-1}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{1-2}-(1)P^{-1}+P^{-2}(1-1)}{[1-P^{-1}]^2}\right]\right\}\right]$$

$$\Rightarrow I_A(1) = \left\{\frac{\alpha_1 P}{P^T}\left[\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right] + \frac{\alpha_2 P}{P^T(1-P^{-1})^2}\left[P^{T-2}-TP^{-1} + P^{-2}(T-1)\right]\right\} - \left\{\frac{\alpha_1 P^{1-1}}{1-P^{-1}} - \frac{\alpha_1 P^{-2}}{1-P^{-1}} - \frac{\alpha_2 P^{-1}}{(1-P^{-1})^2} + \frac{\alpha_2 P^{-1}}{(1-P^{-1})^2} - \frac{\alpha_2 P^{-2}(1-1)}{(1-P^{-1})^2}\right\}$$

$$= \frac{\alpha_1 P}{P^T}\left[\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right] + \frac{\alpha_2 P}{P^T}\left[\frac{[P^{T-2}-TP^{-1}+P^{-2}(T-1)]}{(1-P^{-1})^2}\right] - \frac{\alpha_1}{1-P^{-1}} + \frac{\alpha_1 P^{-2}}{1-P^{-1}} + \frac{\alpha_2 P^{-1}}{(1-P^{-1})^2} - \frac{\alpha_2 P^{-1}}{(1-P^{-1})^2}$$

$at\ t = 2$

$$I_A(2) = \frac{P^2}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \frac{P^2}{P^T}\alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{2-1}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{2-2}-2P^{-1}+P^{-2}(2-1)}{[1-P^{-1}]^2}\right]\right\}\right]$$

$$= \frac{\alpha_1 P^2}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \frac{\alpha_2 P^2}{P^T}\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{1-2P^{-1}+P^{-2}}{[1-P^{-1}]^2}\right]\right\}\right]$$

at $t = 3$

$$I_A(3) = \frac{\alpha_1 P^3}{P^T}\left[\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \frac{\alpha_2 P^3}{P^T}\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{3-1}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{3-2}-3P^{-1}+P^{-2}(3-1)}{[1-P^{-1}]^2}\right]\right\}\right]$$

$$= \frac{\alpha_1 P^3}{P^T}\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \frac{\alpha_2 P^3}{P^T}\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right) + \frac{\alpha_1 P^{-2}}{1-P^{-1}} - \frac{\alpha_1 P^2}{1-P^{-1}} - \frac{\alpha_2 P}{(1-P^{-1})^2} + \frac{3\alpha_2 P^{-1}}{(1-P^{-1})^2} - \frac{2\alpha_2 P^{-2}}{(1-P^{-1})^2}$$

At $t = 4$

$$I_A(4) = \frac{P^4}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{4-1}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{4-2}-4P^{-1}+P^{-2}(4-1)}{[1-P^{-1}]^2}\right]\right\}\right]$$

$$= \frac{\alpha_1 P^4}{P^T}\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \frac{\alpha_2 P^4}{P^T}\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right) + \frac{\alpha_1 P^{-2}}{1-P^{-1}} - \frac{\alpha_1 P^3}{1-P^{-1}} - \frac{\alpha_2 P^2}{(1-P^{-1})^2} + \frac{4\alpha_2 P^{-1}}{(1-P^{-1})^2} - \frac{3\alpha_2 P^{-2}}{(1-P^{-1})^2}$$

At $t = 5$

$$I_A(5) = \frac{P^5}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{5-1}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{5-2}-5P^{-1}+P^{-2}(5-1)}{[1-P^{-1}]^2}\right]\right\}\right]$$

$$= \frac{\alpha_1 P^5}{P^T}\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \frac{\alpha_2 P^5}{P^T}\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right) + \frac{\alpha_1 P^{-2}}{1-P^{-1}} - \frac{\alpha_1 P^4}{1-P^{-1}} - \frac{\alpha_2 P^3}{(1-P^{-1})^2} + \frac{5\alpha_2 P^{-1}}{(1-P^{-1})^2} - \frac{4\alpha_2 P^{-2}}{(1-P^{-1})^2}$$

Continuing in this way we see that
at $t = (T-1)$

$$I_A(T-1) = \frac{P^{T-1}}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{T-2}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{(T-1)-2}-(T-1)P^{-1}+P^{-2}([T-1]-1)}{[1-P^{-1}]^2}\right]\right\}\right]$$

$$= \frac{P^{T-1}}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{(T-2)}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{(T-3)}-(T-1)P^{-1}+P^{-2}([T-2])}{[1-P^{-1}]^2}\right]\right\}\right]$$

$$\text{and at } t = T$$

$$I_A(T) = \frac{P^T}{P^T}\left[\alpha_1\left(\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right) + \alpha_2\left(\frac{P^{T-2}-TP^{-1}+P^{-2}(T-1)}{(1-P^{-1})^2}\right)\right] - \left[\left\{\alpha_1\left[\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right] - \alpha_2\left[\frac{P^{T-2}-TP^{-1}+P^{-2}([T-1])}{[1-P^{-1}]^2}\right]\right\}\right]$$

$\therefore$ the summation, $\sum_{t=0}^{T} I_A(t) =$

$$\left\{\frac{\alpha_1}{P^T}\left[\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right](1+P+P^2+P^3+P^4+P^{T-1}+\cdots+P^T) + \left[\frac{\alpha_2(P^{T-2}-TP^{-1}+(T-1)P^{-2})}{P^T(1-P^{-1})^2}\right](1+P+P^2+P^3+P^4+P^{T-1}+\cdots+P^T)\right\} + \frac{\alpha_1 P^{-2}}{1-P^{-1}}(T+1) - \frac{\alpha_1}{1-P^{-1}}[P^{-1}+1+P+P^2+P^3+P^4+\cdots+P^{T-1}] - \frac{\alpha_2}{(1-P^{-1})^2}\{P^{-2}+P^{-1}+1+P+P^2+P^3+P^4+\cdots+P^{T-3}+P^{T-2}\} + \frac{\alpha_2 P^{-1}}{(1-P^{-1})^2}[0+1+2+3+4+5+\cdots+(T-1)+T] - \frac{\alpha_2 P^{-2}}{(1-P^{-1})^2}[-1+0+1+2+3+4+5+\cdots+(T-2)+(T-1)]$$

$$(i)\ \frac{\alpha_1}{P^T}\left[\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right](1+P+P^2+P^3+P^4+P^{T-1}+\cdots++P^T)$$

Sum of $1+P+P^2+P^3+P^4+P^{T-1}+\cdots++P^T = \left(\frac{P^{T+1}-1}{P-1}\right)$

$$(ii)\ \frac{\alpha_2}{P^T}\left[\frac{(P^{T-2}+(T-1)P^{-2}-TP^{-1})}{(1-P^{-1})^2}\right](1+P+P^2+P^3+P^4+P^{T-1}+\cdots+P^T)$$

Sum of $1+P+P^2+P^3+P^4+P^{T-1}+\cdots+P^T = \left(\frac{P^{T+1}-1}{P-1}\right)$

(iii) $\frac{\alpha_1 P^{-2}}{1-P^{-1}}[T+1]$

$$(iv)\ \frac{\alpha_1}{1-P^{-1}}[P^{-1}+P^0+P+P^2+P^3+\cdots+P^{T-1}]$$

Sum of $P^{-1}+P^0+P+P^2+P^3+\cdots+P^{T-1} = \left(\frac{P^T-P^{-1}}{P-1}\right)$

$$(v)\ \frac{\alpha_2}{(1-P^{-1})^2}[P^{-2}+P^{-1}+P^0+P^1+P^2+P^3+\cdots+P^{T-2}]$$

Sum of $P^{-2}+P^{-1}+P^0+P^1+P^2+P^3+\cdots+P^{T-2} = \left(\frac{P^{T-1}-P^{-2}}{P-1}\right)$

(vi) $-\frac{\alpha_2 P^{-1}}{(1-P^{-1})^2}[0+1+2+3+4+\cdots+(T-1)+T]$

Sum of $0+1+2+3+4+\cdots+(T-1)+T = \frac{T^2+T}{2}$

$$(vii)\ +\frac{\alpha_2 P^{-2}}{(1-P^{-1})^2}[-1+0+1+2+3+4+\cdots+(T-2)+(T-1)]$$

Sum of $-1+0+1+2+3+4+\cdots+(T-2)+(T-1) = \frac{T^2-T-2}{2}$

$$\therefore \sum_{t=0}^{T} I_A(t) = \left\{\frac{\alpha_1}{P^T}\left[\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right]\left(\frac{P^{T+1}-1}{P-1}\right) + \frac{\alpha_2}{P^T}\left[\frac{(P^{T-2}+(T-1)P^{-2}-TP^{-1})}{(1-P^{-1})^2}\right]\left(\frac{P^{T+1}-1}{P-1}\right)\right\} + \frac{\alpha_1 P^{-2}}{1-P^{-1}}[T+1] - \frac{\alpha_1}{1-P^{-1}}\left(\frac{P^T-P^{-1}}{P-1}\right) - \frac{\alpha_2 P^{-2}}{(1-P^{-1})^2}\left(\frac{P^{T-1}-P^{-2}}{P-1}\right) + \frac{\alpha_2 P^{-1}}{(1-P^{-1})^2}\left(\frac{T^2+T}{2}\right) - \frac{\alpha_2}{(1-P^{-1})^2}\left(\frac{T^2-T-2}{2}\right)$$

To calculate the holding cost, we have

$$H_C = i\ \%\ X\ cost\ per\ unit\ X \sum_{t=0}^{T} I_A(t)$$

$$H_C = iC\sum_{t=0}^{T} I_A(t)$$

$$= iC\left\{\left\{\frac{\alpha_1}{P^T}\left[\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right]\left(\frac{P^{T+1}-1}{P-1}\right) + \frac{\alpha_2}{P^T}\left[\frac{(P^{T-2}+(T-1)P^{-2}-TP^{-1})}{(1-P^{-1})^2}\right]\left(\frac{P^{T+1}-1}{P-1}\right)\right\} + \frac{\alpha_1 P^{-2}}{1-P^{-1}}[T+1] - \frac{\alpha_1}{1-P^{-1}}\left(\frac{P^T-P^{-1}}{P-1}\right) - \frac{\alpha_2 P^{-2}}{(1-P^{-1})^2}\left(\frac{P^{T-1}-P^{-2}}{P-1}\right) + \frac{\alpha_2 P^{-1}}{(1-P^{-1})^2}\left(\frac{T^2+T}{2}\right) - \frac{\alpha_2}{(1-P^{-1})^2}\left(\frac{T^2-T-2}{2}\right)\right\} \quad (8)$$

The total demand in the interval $(0 \le t \le T)$ is given by

$$D_T = \sum_{t=0}^{T}(\alpha_1+\alpha_2 t) \Rightarrow \sum_{t=0}^{T}\alpha_1 + \alpha_2\sum_{t=0}^{T}(t)$$

$$= \alpha_1(T+1) + \alpha_2(0+1+2+3+4+5+\cdots+(T-1)+T)$$

$$\therefore\ D_T = \sum_{t=0}^{T}(\alpha_1+\alpha_2 t) = \alpha_1(T+1) + \alpha_2\left(\frac{T^2+T}{2}\right)$$

Amount due to Amelioration is expressedas

$A_m = D_T - I_A(0)$ i.e total demand – order quantity

$\boldsymbol{A_m} = \sum_{t=0}^{T}(\alpha_1+i\alpha_2) - \frac{\sum_{i-0}^{T-1}(1+A)^{T-1-i}[\alpha_1+i\alpha_2]}{(1+A)^T}$From equation (4)

$$= \alpha_1(T+1) + \alpha_2\left(\frac{T^2+T}{2}\right) - \frac{\sum_{i-0}^{T-1}(1+A)^{T-1-i}[\alpha_1+i\alpha_2]}{(1+A)^T}$$

$$= \alpha_1(T+1) + \alpha_2\left(\frac{T^2+T}{2}\right) - (1+A)^{-T}\alpha_1\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right] + (1+A)^{-T}\alpha_2\left[\frac{(1+A)^{T-2}-T(1+A)^{-1}+(T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]$$

The cycle's total variable cost (TVC) is formulated as TVC = Ordering cost + holding cost – cost of ameliorated amount

$$= C_0 + H_C - CA_m$$

$$TVC = C_0 + iC\left\{\left\{\frac{\alpha_1}{P^T}\left[\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right]\left(\frac{P^{T+1}-1}{P-1}\right) + \frac{\alpha_2}{P^T}\left[\frac{(P^{T-2}+(T-1)P^{-2}-TP^{-1})}{(1-P^{-1})^2}\right]\left(\frac{P^{T+1}-1}{P-1}\right)\right\} + \frac{\alpha_1 P^{-2}}{1-P^{-1}}[T+1] - \frac{\alpha_1}{1-P^{-1}}\left(\frac{P^T-P^{-1}}{P-1}\right) - \frac{\alpha_2 P^{-2}}{(1-P^{-1})^2}\left(\frac{P^{T-1}-P^{-2}}{P-1}\right) + \frac{\alpha_2 P^{-1}}{(1-P^{-1})^2}\left(\frac{T^2+T}{2}\right) - \frac{\alpha_2}{(1-P^{-1})^2}\left(\frac{T^2-T-2}{2}\right)\right\} - C\left\{\alpha_1(T+1) + \alpha_2\left(\frac{T^2+T}{2}\right) - P^{-T}\alpha_1\left[\frac{P^{T-1}-P^{-2}}{1-P^{-1}}\right] + P^{-T}\alpha_2\left[\frac{P^{T-2}-TP^{-1}+(T-1)P^{-2}}{[1-P^{-1}]^2}\right]\right\} \quad (9)$$

Recall that P $= (1+A)$, so if we revert back to (1+A) for P,

The average variable cost per unit time is

$$= \left\{\frac{C_0}{T} + \frac{iC}{T+1}\left\{\left\{\frac{\alpha_1}{(1+A)^T}\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\left(\frac{(1+A)^{T+1}-1}{(1+A)-1}\right) + \frac{\alpha_2}{(1+A)^T}\left[\frac{((1+A)^{T-2}+(T-1)(1+A)^{-2}-T(1+A)^{-1})}{(1-(1+A)^{-1})^2}\right]\left(\frac{(1+A)^{T+1}-1}{(1+A)-1}\right)\right\} + \frac{\alpha_1(1+A)^{-2}}{1-(1+A)^{-1}}[T+1] - \frac{\alpha_1}{1-(1+A)^{-1}}\left(\frac{(1+A)^T-(1+A)^{-1}}{(1+A)-1}\right) - \frac{\alpha_2(1+A)^{-2}}{(1-(1+A)^{-1})^2}\left(\frac{(1+A)^{T-1}-(1+A)^{-2}}{(1+A)-1}\right) + \frac{\alpha_2(1+A)^{-1}}{(1-(1+A)^{-1})^2}\left(\frac{T^2+T}{2}\right) - \frac{\alpha_2}{(1-(1+A)^{-1})^2}\left(\frac{T^2-T-2}{2}\right)\right\} - \frac{C}{T+1}\left\{\alpha_1(T+1) + \alpha_2\left(\frac{T^2+T}{2}\right) - (1+A)^{-T}\alpha_1\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right] + (1+A)^{-T}\alpha_2\left[\frac{(1+A)^{T-2}-T(1+A)^{-1}+(T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]\right\}\right\} \quad (10)$$

In the above equation C_0, which is the ordering cost is divided by (T), while the holding cost and the cost of ameliorated amount is divided by $(T+1)$ because the summation for their values is from zero to T, hence we have $(T+1)$ terms.

In a similar way, for TVC (T-1), let T-1=s so that TVC (T-1) = TVC (s) then

$$
\begin{aligned}
TVC\ (s) = \frac{C_0}{s} + \frac{iC}{T+1}\Bigg\{\Bigg\{\frac{\alpha_1}{(1+A)^s}\left[\frac{(1+A)^{s-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\left(\frac{(1+A)^{s+1}-1}{(1+A)-1}\right) \\
+ \frac{\alpha_2}{(1+A)^s}\left[\frac{((1+A)^{s-2}+(s-1)(1+A)^{-2}-s(1+A)^{-1})}{(1-(1+A)^{-1})^2}\right]\left(\frac{(1+A)^{s+1}-1}{(1+A)-1}\right)\Bigg\} \\
+ \frac{\alpha_1(1+A)^{-2}}{1-(1+A)^{-1}}[s+1] - \frac{\alpha_1}{1-(1+A)^{-1}}\left(\frac{(1+A)^s-(1+A)^{-1}}{(1+A)-1}\right) \\
- \frac{\alpha_2(1+A)^{-2}}{(1-(1+A)^{-1})^2}\left(\frac{(1+A)^{s-1}-(1+A)^{-2}}{(1+A)-1}\right) + \frac{\alpha_2(1+A)^{-1}}{(1-(1+A)^{-1})^2}\left(\frac{s^2+s}{2}\right) \\
- \frac{\alpha_2}{(1-(1+A)^{-1})^2}\left(\frac{s^2-s-2}{2}\right)\Bigg\} \\
- \frac{C}{T+1}\Bigg\{\alpha_1(s+1) + \alpha_2\left(\frac{s^2+s}{2}\right) - (1+A)^{-s}\alpha_1\left[\frac{(1+A)^{s-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right] \\
+ (1+A)^{-s}\alpha_2\left[\frac{(1+A)^{s-2}-s(1+A)^{-1}+(s-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]\Bigg\}
\end{aligned}
$$

Note that conversion of (T-1) to (s) is done to represent a
unit less than T, which may or may not be equal to (T-1)
Also, for TVC (T+1), let T+1 = e, so that TVC (T+1) = TVC (e)

$$
\begin{aligned}
TVC\ (e) = \frac{C_0}{e} + \frac{iC}{T+1}\Bigg\{\Bigg\{\frac{\alpha_1}{(1+A)^e}\left[\frac{(1+A)^{e-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\left(\frac{(1+A)^{e+1}-1}{(1+A)-1}\right) \\
+ \frac{\alpha_2}{(1+A)^e}\left[\frac{((1+A)^{e-2}+(e-1)(1+A)^{-2}-e(1+A)^{-1})}{(1-(1+A)^{-1})^2}\right]\left(\frac{(1+A)^{e+1}-1}{(1+A)-1}\right)\Bigg\} \\
+ \frac{\alpha_1(1+A)^{-2}}{1-(1+A)^{-1}}[e+1] - \frac{\alpha_1}{1-(1+A)^{-1}}\left(\frac{(1+A)^e-(1+A)^{-1}}{(1+A)-1}\right) \\
- \frac{\alpha_2(1+A)^{-2}}{(1-(1+A)^{-1})^2}\left(\frac{(1+A)^{e-1}-(1+A)^{-2}}{(1+A)-1}\right) + \frac{\alpha_2(1+A)^{-1}}{(1-(1+A)^{-1})^2}\left(\frac{e^2+e}{2}\right) \\
- \frac{\alpha_2}{(1-(1+A)^{-1})^2}\left(\frac{e^2-e-2}{2}\right)\Bigg\} \\
- \frac{C}{T+1}\Bigg\{\alpha_1(e+1) + \alpha_2\left(\frac{e^2+e}{2}\right) - (1+A)^{-e}\alpha_1\left[\frac{(1+A)^{e-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right] \\
+ (1+A)^{-e}\alpha_2\left[\frac{(1+A)^{e-2}-e(1+A)^{-1}+(e-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]\Bigg\}
\end{aligned}
$$

As earlier commented, the conversion of (T+1) to (e) is also done to represent a unit more than T, which may or may not be equal to (T+1)

TVC (T) – TVC (s)

$$
\begin{aligned}
&= \frac{C_0(s-T)}{Ts} + \frac{iC}{T+1}\left\{\frac{\alpha_1}{(1+A)^T}\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\left(\frac{(1+A)^{T+1}-1}{(1+A)-1}\right)\right.\\
&\qquad \left. -\ \frac{\alpha_1}{(1+A)^s}\left[\frac{(1+A)^{s-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\left(\frac{(1+A)^{s+1}-1}{(1+A)-1}\right)\right\}\\
&\qquad + \frac{iC}{T+1}\left\{\frac{\alpha_2}{(1+A)^T}\left[\frac{((1+A)^{T-2}+(T-1)(1+A)^{-2}-T(1+A)^{-1})}{(1-(1+A)^{-1})^2}\right]\left(\frac{(1+A)^{T+1}-1}{(1+A)-1}\right)\right.\\
&\qquad - \frac{\alpha_2}{(1+A)^s}\left[\frac{((1+A)^{s-2}+(s-1)(1+A)^{-2}-s(1+A)^{-1})}{(1-(1+A)^{-1})^2}\right]\left(\frac{(1+A)^{s+1}-1}{(1+A)-1}\right)\\
&\qquad + \frac{\alpha_1(1+A)^{-2}}{1-(1+A)^{-1}}[T+1] - \frac{\alpha_1(1+A)^{-2}}{1-(1+A)^{-1}}[s+1] - \frac{\alpha_1}{1-(1+A)^{-1}}\left(\frac{(1+A)^T-(1+A)^{-1}}{(1+A)-1}\right)\\
&\qquad + \frac{\alpha_1}{1-(1+A)^{-1}}\left(\frac{(1+A)^s-(1+A)^{-1}}{(1+A)-1}\right) - \frac{\alpha_2(1+A)^{-2}}{(1-(1+A)^{-1})^2}\left(\frac{(1+A)^{T-1}-(1+A)^{-2}}{(1+A)-1}\right)\\
&\qquad + \frac{\alpha_2(1+A)^{-2}}{(1-(1+A)^{-1})^2}\left(\frac{(1+A)^{s-1}-(1+A)^{-2}}{(1+A)-1}\right) + \frac{\alpha_2(1+A)^{-1}}{(1-(1+A)^{-1})^2}\left(\frac{T^2+T}{2}\right)\\
&\qquad - \frac{\alpha_2(1+A)^{-1}}{(1-(1+A)^{-1})^2}\left(\frac{s^2+s}{2}\right) - \frac{\alpha_2}{(1-(1+A)^{-1})^2}\left(\frac{T^2-T-2}{2}\right)\\
&\qquad \left. + \frac{\alpha_2}{(1-(1+A)^{-1})^2}\left(\frac{s^2-s-2}{2}\right)\right\}\\
&\qquad - \frac{C}{T+1}\left\{\alpha_1(T+1) - \alpha_1(s+1) + \alpha_2\left(\frac{T^2+T}{2}\right) - \alpha_2\left(\frac{s^2+s}{2}\right)\right.\\
&\qquad - (1+A)^{-T}\alpha_1\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right] + (1+A)^{-s}\alpha_1\left[\frac{(1+A)^{s-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\\
&\qquad + (1+A)^{-T}\alpha_2\left[\frac{(1+A)^{T-2}-T(1+A)^{-1}+(T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]\\
&\qquad \left. - (1+A)^{-s}\alpha_2\left[\frac{(1+A)^{s-2}-s(1+A)^{-1}+(s-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]\right\} \qquad (11)
\end{aligned}
$$

Similarly, TVC (e) – TVC (T) =

$$
\begin{aligned}
&\frac{C_0(T-e)}{eT}+\frac{iC}{T+1}\left\{\frac{\alpha_1}{(1+A)^e}\left[\frac{(1+A)^{e-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\left(\frac{(1+A)^{e+1}-1}{(1+A)-1}\right)\right.\\
&\qquad\left.-\frac{\alpha_1}{(1+A)^T}\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\left(\frac{(1+A)^{T+1}-1}{(1+A)-1}\right)\right\}\\
&\qquad+\frac{iC}{T+1}\left\{\frac{\alpha_2}{(1+A)^e}\left[\frac{((1+A)^{e-2}+(e-1)(1+A)^{-2}-e(1+A)^{-1})}{(1-(1+A)^{-1})^2}\right]\left(\frac{(1+A)^{e+1}-1}{(1+A)-1}\right)\right.\\
&\qquad-\frac{\alpha_2}{(1+A)^T}\left[\frac{((1+A)^{T-2}+(T-1)(1+A)^{-2}-T(1+A)^{-1})}{(1-(1+A)^{-1})^2}\right]\left(\frac{(1+A)^{T+1}-1}{(1+A)-1}\right)\\
&\qquad+\frac{\alpha_1(1+A)^{-2}}{1-(1+A)^{-1}}[e+1]-\frac{\alpha_1(1+A)^{-2}}{1-(1+A)^{-1}}[T+1]-\frac{\alpha_1}{1-(1+A)^{-1}}\left(\frac{(1+A)^{e}-(1+A)^{-1}}{(1+A)-1}\right)\\
&\qquad+\frac{\alpha_1}{1-(1+A)^{-1}}\left(\frac{(1+A)^{T}-(1+A)^{-1}}{(1+A)-1}\right)-\frac{\alpha_2(1+A)^{-2}}{(1-(1+A)^{-1})^2}\left(\frac{(1+A)^{e-1}-(1+A)^{-2}}{(1+A)-1}\right)\\
&\qquad+\frac{\alpha_2(1+A)^{-2}}{(1-(1+A)^{-1})^2}\left(\frac{(1+A)^{T-1}-(1+A)^{-2}}{(1+A)-1}\right)+\frac{\alpha_2(1+A)^{-1}}{(1-(1+A)^{-1})^2}\left(\frac{e^2+e}{2}\right)\\
&\qquad-\frac{\alpha_2(1+A)^{-1}}{(1-(1+A)^{-1})^2}\left(\frac{T^2+T}{2}\right)-\frac{\alpha_2}{(1-(1+A)^{-1})^2}\left(\frac{e^2-e-2}{2}\right)\\
&\qquad\left.+\frac{\alpha_2}{(1-(1+A)^{-1})^2}\left(\frac{T^2-T-2}{2}\right)\right\}\\
&\qquad-\frac{C}{T+1}\left\{\alpha_1(e+1)-\alpha_1(T+1)+\alpha_2\left(\frac{e^2+e}{2}\right)-\alpha_2\left(\frac{T^2+T}{2}\right)\right.\\
&\qquad-(1+A)^{-e}\alpha_1\left[\frac{(1+A)^{e-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]+(1+A)^{-T}\alpha_1\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\\
&\qquad+(1+A)^{-e}\alpha_2\left[\frac{(1+A)^{e-2}-e(1+A)^{-1}+(e-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]\\
&\qquad\left.-(1+A)^{-T}\alpha_2\left[\frac{(1+A)^{T-2}-T(1+A)^{-1}+(T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]\right\} \qquad (12)
\end{aligned}
$$

Optimality Condition

The optimality conditions for the value of T to minimize TVC (T) are

$TVC\ (T^*) \le TVC\ (s)$ and $TVC(T^*) \le TVC$ (e)($T = T^* \ \ge 0$)

$\Rightarrow TVC\ (T^*) - TVC$ (s) ≤ 0 and TVC (e) $- TVC\ (T^*) \ge 0$

$\Rightarrow \Delta TVC$ (s) ≤ 0 and $\Delta TVC\ (T^*) \ge 0$

Thus

$$\Delta TVC\ (\text{s}) \le 0 \le \Delta TVC\ (T^*)$$

Computation of the EOQ

EOQ is defined as the total cycle demand minusthe ameliorated amounti.e

EOQ = D_T - A_m

$$= \sum_{t=0}^{T}(\alpha_1 + i\alpha_2) - \left[\alpha_1(T+1) + \alpha_2\left(\frac{T^2+T}{2}\right) - \frac{\sum_{i=0}^{T-1}(1+A)^{T-1-i}[\alpha_1+i\alpha_2]}{(1+A)^T}\right]$$

$$
\begin{aligned}
&= \alpha_1(T+1) + \alpha_2\left(\frac{T^2+T}{2}\right) - \alpha_1(T+1)\\
&\qquad - \alpha_2\left(\frac{T^2+T}{2}\right)\\
&\qquad -(1+A)^{-T}\alpha_1\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\\
&\qquad + (1 + A)^{-T}\alpha_2\left[\frac{(1+A)^{T-2}-T(1+A)^{-1}+(T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right]
\end{aligned}
$$

On simplification, we have

$$
\begin{aligned}
EOQ &= -(1+A)^{-T}\alpha_1\left[\frac{(1+A)^{T-1}-(1+A)^{-2}}{1-(1+A)^{-1}}\right]\\
&\quad + (1 + A)^{-T}\alpha_2\left[\frac{(1+A)^{T-2}-T(1+A)^{-1}+(T-1)(1+A)^{-2}}{[1-(1+A)^{-1}]^2}\right] \qquad (13)
\end{aligned}
$$

RESULTS AND DISCUSSION

The mathematical model has been formed that give the optimality condition. Likewise, the EOQ has also been calculated. Excel software was used to get the best T that satisfies the optimality condition using numerical example.

Numerical example

An example is considered using the following parameters: $C_0 = 2000$,i = 0.30, $\alpha_1 = 20000, \alpha_2 = 2.0$, C = 300, A = 0.20. Using equations (10), (11), (12) and

(13). The optimal results gotten are as follows: T = 33, TVC = 51079.63 and EOQ = 1634.97.

Sensitivity Analysis

The example above is usedto carry out a sensitivity analysis to check the behaviour of the decision variablesT, TVC (T) and EOQ. This is done by varying each of the parameter by +50%, +25%, +10%, +5%, -5%, -10%, -25% and - 50% at a time and keeping others unchanged.

Table: 1 **Effect of the Sensitivity Analysis in Decision Variables T, TVC (T) and EOQ.**

Parameter	**% Change**	**Change in Results**		
		T^*(DAYS)	**TVC(T^*)**	**EOQ^***
C_0	+50	41	60958.33	2026.84
	+25	37	56265.33	1831.11
	+10	35	53223.25	1733.09
	+5	34	52163.42	1684.04
	0	**33**	**51079.63**	**1634.97**
	-5	32	49969.62	1585.88
	-10	32	48828.99	1585.88
	-25	29	45196.35	1438.43
	-50	24	38228.82	1192.19
I	+50	21	82391.58	1044.15
	+25	25	68225.27	1241.49
	+10	29	58450.28	1438.44
	+5	31	54871.85	1536.76
	0	**33**	**51079.63**	**1634.97**
	-5	36	47018.50	1782.11
	-10	40	42612.49	1977.94
	-25	70	25395.35	
	-50	3433.92	NO SOLUTION	
C	+50	27	64533.59	1340.02
	+25	30	58054.65	1487.61
	+10	32	53940.02	1585.88
	+5	32	52525.14	1585.88
	0	**33**	**51079.63**	**1634.97**
	-5	34	49608.93	1684.04
	-10	35	48109.50	1733.09
	-25	38	43409.17	1880.08
	-50	47	34645.93	2319.68
α_1	+50	27	65811.32	1943.89
	+25	29	61799.37	2009.83
	+10	32	53422.94	1744.45
	+5	32	52266.59	1665.16

	0	**33**	**51079.63**	**1634.97**
	-5	34	49861.68	1599.84
	-10	35	48609.27	1559.78
	-25	39	44606.26	1446.71
	-50	49	36795.52	1208.15
A	50	No solution		
	25	48	35386.78	2313.20
	10	37	45192.34	1814.18
	5	35	48174.22	1725.07
	0	**33**	**51079.63**	**1634.97**
	-5	32	53934.43	1593.24
	-10	30	56763.90	1501.44
	-25	28	65319.82	1421.88
	-50	27	80421.03	1404.21
α_2	+50	34	53617.41	1684.07
	+25	34	52353.65	1684.05
	+10	33	51590.91	1634.98
	+5	33	51335.27	1634.98
	0	**33**	**51079.63**	**1634.97**
	-5	33	50824.00	1634.97
	-10	33	50568.36	1634.96
	-25	33	49801.45	1634.95
	-50	33	48523.27	1634.92

Discussion of Results

From table 1 above, the following are the observation made about the sensitivity analysis.

(i) Whenthe ordering cost,C_0 increases, EOQ, TVC and T increases. Justification to this is that, ahigher ordering cost leads to placing orders less often and larger each time which lead to high EOQ and high EOQ lead to high TVC and T as they have direct proportionality effect.

(ii) When the carrying charge, i increases, TVC increases but EOQ and T decreases. Explanations to this is that a higher carrying charge leads to smaller lots (EOQ). Smaller EOQ leads to shorter cycle time (T) but TVC is high because carrying charge per unit is high.

(iii) IfC, (cost of the item) increases, TVC decreases, EOQ increases and T also increases. When there is high cost of the item, It is expected that both EOQ and T will decrease. However, the model is trying to reduce cost, and this probably increases both the EOQ and the T,

(iv) When the demand, α_1 increases,EOQ and TVC increases but T decreases. The EOQ increases because more items are needed to cater for the large demand. The TVC increases because of the increase in EOQ. T reduces as a result of increase in demand.

(v) As the amelioration A, increases, EOQ and T increases but TVC decreases. The reason is that when amelioration increases more items are purchased to stock so as to take the advantage of increase in amelioration which then increases the EOQ. Increasedameliorationhelps to reduce cost and hence TVC will reduce. T increases due to increase in EOQ. These are all expected

With increase in α_2, the EOQ decreases. The TVC increases but the T is not affected by either increase or decrease, because the demand in this case is dependent on T.

CONCLUSION

In this research, a discrete time EOQ model for ameliorating items with constant holding cost and linear demand is presented. Items that display ameliorating behaviour includes; chickens, fish, ducks, cows, sheep, and others. The inventory begins with products purchased from outside and placed in stock. The items begin to improve, and as they reach their peak level of improvement, the stock decreases due to demand alone. Our goal is to discover the optimal replenishment cycle that minimizes overall variable cost. The model put into consideration the effect of amelioration on inventory, which partially offsets depletion due to demand. Also the consideration of linear demand provides a more realistic situation of consumption patterns compared to constant demand, thereby allowing for improved accuracy in determining order quantity and cycle length. A numerical example was provided to demonstrate the model's application, and a sensitivity analysis was performed to assess the impact of parameter changes.

The classical EOQ model typically assumes that time is continuous, however, in many real-world scenarios, especially with periodic review policies, time is better represented as a sequence of discrete intervals. This means that inventory information and decisions are made at the end of each period, the discrete-time EOQ models modifies the original EOQ assumptions to reflect this reality. Further research can be considered with linear holding cost, shortages, backlogging just to mention a few.

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