



## On Permuting $Z_4$ - $f$ and $Z_4$ -( $f, g$ )- Derivations of Lattices

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### ABSTRACT

The aim of this paper is to extend results on derivation of lattices by introducing permuting  $Z_4$ - $f$  and  $Z_4$ -( $f, g$ )-derivations. The ideas of modular identity, isotone derivations and trace of permuting  $Z_4$ - $f$  and  $Z_4$ -( $f, g$ )-derivations are investigated. Conditions that establish the distributive property and other related properties are studied, and new results are presented.

### INTRODUCTION

Lattice theory made tremendous developments in the last decades. The notion of derivation appeared first on ring structures. For instance, Posner (1988) introduced the notion of derivation on a prime ring  $R$  as a function  $d$  from  $R$  into itself satisfying the following conditions  $d(x, y) = d(x, y) + xd(y)$  and  $d(x + y) = d(x) + d(y)$  for all  $x, y \in R$ . The notion of derivation on rings has many applications. Szász (1975) extended the concept of derivation to lattice structures based on the meet and the join operations ( $\wedge, \vee$ )-derivation. Raji (1988) presented equivalent conditions for a derivation to be isotone in specific types of lattices with greatest elements, the author however characterized modular lattices and distributive lattices using the properties of derivations. Farrari (2001) studied lattices that are either unbounded or non distributive which are the only cases in which the derivations can be nontrivial. The author also considered the lattice  $[N; \max, \min]$  whose associated partially ordered set (poset) is the chain  $[N; \leq]$  with the usual total order. Xin et al (2008) studied the notion of derivation on a lattice by considering the first condition of Posner's theorem (Posner, 1988). It has been shown that the second condition holds for isotone derivations on a distributive lattice. In the same paper, the authors characterized the distributive and modular lattices in terms of isotone derivations. Ceven & Ozturk (2008) gave generalization of derivation of a lattice which was defined in Xin et al (2008). Ceven (2009) introduced the symmetric classical-derivation and  $f$ -derivations of lattices. Alshehli (2010) used the idea of isotone generalized derivation and gave characterizations of modular lattices and distributive lattices. Ozbal & Firat (2010) introduced the notion of  $f$  bi-derivation on a lattice they also characterized distributive lattices using symmetric  $f$  bi-derivation.

Mustafa et al (2011) gave some equivalent conditions under which a derivation is isotone for lattices with a greatest element, modular lattices and distributive lattices. The authors investigated some related properties of symmetric bi-( $\sigma, \tau$ )-derivations in near rings, characterized the distributive and modular lattices by the trace. Yazarli & Ozturk (2011) introduced the notion of permuting tri- $f$  and tri-( $f, g$ )-derivations in lattices and studied some of its properties. Mustafa & Sahin (2013) characterized distributive lattices and isotone derivation using generalized ( $f, g$ )-derivations and gave illustrative examples. The authors also generalized the derivation defined in Ceven et al. Balogun (2014) introduced a derivation on a lattice  $L$ ;  $f(x \cap y) = x \cap fy$ . The author used this notion to establish equivalence relation on  $L$ . Some results on isotone derivations on distributive lattices were also extended. Ceven (2019) introduced the notion of higher derivation on a lattice. Leerawat et al (2022) introduced the concept of trace permuting  $n$  - ( $f, g$ ) - derivation of lattice and presented some related properties. By proposing new parameters under well-defined constraints, new results on derivations on lattices can be obtained for further characterizations. In this paper, results on lattice derivation will be extended by introducing new parameters for permuting tri- $f$ , tri-( $f, g$ ),  $Z_4$ - $f$  and  $Z_4$ -( $f, g$ )-derivations on lattices.

### Preliminaries

In this section, definitions that will be used in this study are presented.

Definition 2.1: Let  $L$  be a non- empty set together with two binary operations  $\cap$  and  $\cup$ , if the algebraic structure  $(L, \cap, \cup)$  satisfies the following conditions

- (i)  $x \cap x = x, x \cup x = x$
- (ii)  $x \cap y = y \cap x, x \cup y = y \cup x$
- (iii)  $(x \cap y) \cap z = x \cap (y \cap z), (x \cup y) \cup z = x \cup (y \cup z)$

(iv)  $(x \cup y) \cup x = x$ ,  $(x \cup y) \cap x = x$  for every  $x, y, z \in L$

Then  $L$  is called a lattice.

Definition 2.2: A lattice  $(L, \cap, \cup)$  is called a distributive lattice if it satisfies any of the following

(v)  $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$

(vi)  $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$

In any lattice, conditions v and vi are equivalent

Definition 2.3: A lattice  $(L, \cap, \cup)$  is called a modular lattice if it satisfies the following conditions for all  $x, y, z \in L$

(vii) If  $x \leq z$  then  $x \cup (y \cap z) = (x \cup y) \cap z$

Condition vii is called the modular identity

Definition 2.4: Let  $(L, \cap, \cup)$  be a lattice. A binary relation  $\leq$  is defined by  $x \leq y$  if and only if  $x \cap y = x$  and  $x \cup y = y$

Definition 2.5: Let  $L$  be a lattice. Define the binary relation  $\leq$  as in definition 2.4,  $(L, \leq)$  is a partial ordered (poset) and if for any  $x, y \in L$ ,  $x \cap y$  is the greatest lower bound (g.l.b) of  $\{x, y\}$  and  $x \cup y$  is the least upper bound (l.u.b) of  $\{x, y\}$

Definition 2.6: Let  $(L, \cap, \cup)$  be a lattice. A non- empty subset  $I$  is called a left (right) ideal of  $L$  provided the following holds

(i)  $x \leq y, y \in I \rightarrow x \in I$  (ii)  $x, y \in I \rightarrow x \cup y \in I$

$I_1, I_2$  are ideals of  $L$ , implies that  $I_1 \cap I_2$  is an ideal of  $L$

Definition 2.7: Let  $L$  and  $S$  be two lattices. The function  $\theta: L \rightarrow S$  is called a joint and meet homomorphism provided the following holds for all  $x, y \in L$ .

(i)  $\theta(x \cap y) = \theta(x) \cap \theta(y)$  and (ii)  $\theta(x \cup y) = \theta(x) \cup \theta(y)$ ,

hence a homomorphism from  $L$  to  $S$  is isomorphism if  $\theta$  is bijective.

Definition 2.8: A function  $D: L \rightarrow L$  is called derivation of  $L$  if for all  $x, y \in L$

$$\begin{aligned} D(x \cup y) &= D(x) \cup D(y) \\ D(x \cap y) &= D(x) \cap D(y) \end{aligned}$$

Definition 2.9: Let  $L$  be a lattice and  $D$  a derivation on  $L$ , a mapping  $D: L \rightarrow L$  defined by  $d(x) = D(x)$  is called the trace of  $D$ .

Definition 2.10: Let  $L$  be a lattice and  $D$  a derivation on  $L$ , if  $x \leq y$  implies that  $Dx \leq Dy$  then  $D$  is called an isotone derivation, for all  $x, y \in L$ .

Definition 2.11: A function  $D: L \rightarrow L$  is called an  $f$ -derivation on  $L$ , if  $D(x \cap y) = (D(x) \cap f(y)) \cup (f(x) \cap D(y))$ .

Definition 2.12: Let  $L$  be a lattice and  $X$  be a non - empty set, we say  $L$  acts on  $X$  if there exists homomorphism  $D: L \rightarrow S_*$  (the symmetric group  $X$ )

Definition 2.13: Let  $L$  be a lattice. A mapping  $D: L \times L \times L \rightarrow L$  is called permuting if it satisfies the following conditions.

$$D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) =$$

$$D(z, x, y) = D(z, y, x), \text{ for all } x, y, z \in L.$$

If  $D$  is a permuting tri-derivation then  $D$  satisfies the relations

$$(i) \quad D(x, y \cap w, z) = (D(x, y, z) \cap w) \cup (y \cap D(x, w, z))$$

$$(ii) \quad D(x, y, z \cap w) = (D(x, y, z) \cap w) \cup (z \cap D(x, y, w)) \quad \text{For all } w, x, y, z \in L$$

Definition 2.14: Let  $L$  be a lattice. A permuting mapping  $D: L \times L \times L \rightarrow L$  is called permuting tri- $f$ -derivation if there exists a function  $f: L \rightarrow L$  such that

$$D(x \cap w, y, z) = (D(x, y, z) \cap f(x)) \cup (f(x) \cap D(w, y, z))$$

$$\text{For all } w, x, y, z \in L$$

Definition 2.16: Let  $L$  be a lattice and  $D: L \times L \times L \rightarrow L$  be a permuting mapping, then  $D$  is called permuting tri - ( $f, g$ )-derivation of  $L$  if there exists functions  $f, g: L \rightarrow L$  such that

$$\begin{aligned} D(x \cap w, y, z) &= (D(x, y, z) \cap f(w)) \cup (g(x) \cap D(w, y, z)) \\ &\quad \forall w, x, y, z \in L. \end{aligned}$$

## MATERIALS AND METHODS

### Derivation

In this section, results that will be extended using principal derivation, permuting tri -  $f$  and tri - ( $f, g$ ) - derivations of lattices are presented. Firstly, results from Balogun (2014) will be extended to derivation actions of lattices, while results from Yazarli & Ozturk (2011) and Kacilioglu et al (2011) will be extended to permuting  $Z_4$ - $f$  and  $Z_4$ -( $f, g$ ) - derivations of lattices respectively.

Proposition 3.1.1

Let  $L$  be a lattice and  $a \in L$  Define a function  $f_a$  by  $f_a x = x \cup a \quad \forall a \in L$  then  $f$  is a derivation (Principal derivation) on  $L$ .

Proposition 3.1.2

Let  $L$  be a distributive lattice and  $f_1, f_2$  and  $f_3$  be isotone derivations on  $L$ . Define

$$((f_1 \cap f_2) \cap f_3)x = (f_1(x) \cap f_2(x)) \cap f_3(x)$$

$$((f_1 \cup f_2) \cup f_3)x = (f_1(x) \cup f_2(x)) \cup f_3(x), \text{ then}$$

$(f_1 \cap f_2) \cup f_3$  and  $(f_1 \cup f_2) \cap f_3$  are also isotone derivation on  $L$

### 3.2 Permuting tri - $f$ and tri $(f, g)$ – Derivations of lattices

**Definition 1:** Let  $L$  be a lattice. A permuting mapping  $D: L \times L \times L \rightarrow L$  is called permuting tri- $f$ -derivation if there exists a function  $f: L \rightarrow L$  such that  
 $D(x \cap w, y, z) = (D(x, y, z) \cap f(w)) \cup (f(x) \cap D(w, y, z))$  and called permuting tri -  $(f, g)$  – derivation if there exists functions  $f, g: L \rightarrow L$  such that  
 $D(x \cap w, y, z) = (D(x, y, z) \cap f(w)) \cup (g(x) \cap D(w, y, z))$  for all  $x, y, w, z \in L$

The following results in Yazarli & Ozturk (2011) will be extended to permuting  $Z_4$  -  $f$  – derivation of lattices.

#### Proposition 3.2.1

For any  $x \in L$ ,  $D(x, x, x) = d(x) \leq f(x)$  where  $d(x) = D(x \cap x, x, x)$

#### Proposition 3.2.2

Let  $L$  be a lattice and  $D$  be a permuting tri- $f$ -derivation on  $L$ , then the following holds for every  $x, y, z, w \in L$

- (i)  $D(x, y, z) \leq f(x)$ .  $D(x, y, z) \leq f(y)$  and  $D(x, y, z) \leq f(z)$
- (ii)  $D(x, y, z) \cap D(w, y, z) \leq D(x \cap w, y, z) \leq D(x, y, z) \cup D(w, y, z)$
- (iii)  $D(x \cap w, y, z) \leq f(x) \cup f(w)$

#### Proposition 3.2.4

Let  $L$  be a lattice and  $D$  be a permuting tri- $f$ -derivation on  $L$ . If  $L$  is an increasing function then  $w \leq x$  and  $D(x, y, z) = f(x)$  implies that  $D(w, y, z) \leq f(w)$

#### Proposition 3.2.5

Let  $L$  be a distributive lattice and  $D$  a permuting tri- $f$  derivation on  $L$  where  $f(x \cup w) = f(x) \cup f(w)$ , then the following holds.

- (i) If  $D$  is isotone derivation, then  $D(x \cap w, y, z) = D(x, y, z) \cap D(w, y, z)$
- (ii)  $D$  is isotone derivation if and only if  $D(x \cup w, y, z) = D(x, y, z) \cup D(w, y, z)$

The following results in Kacilioglu et al (2011) will be extended to permuting  $Z_4$  -  $(f, g)$  – derivations of lattices.

#### Proposition 3.2.6

Let  $L$  be a lattice and  $D$  be the trace of permuting tri- $(f, g)$ -derivation on  $L$ , then

$$D(x) \leq (f(x) \cup g(x)) \quad \text{for all } x \in L$$

#### Proposition 3.2.7

Let  $L$  be a lattice and  $D$  be a permuting tri- $(f, g)$ -derivation on  $L$ , then

- (i)  $D(x, y, z) \leq f(x) \cup g(x)$  (ii)  $D(x, y, z) \leq f(y) \cup g(y)$  (iii)  $D(x, y, z) \leq f(z) \cup g(z)$
- $f(0) = 0$  and  $g(0) = 0$  then  $D(0, y, z) = 0$  for all  $y, z \in L$

#### Proposition 3.2.9

Let  $L$  be a lattice with a greatest element 1 and  $D$  be a permuting tri- $(f, g)$ -derivation on  $L$ , then following are valid

- (i) If  $f(x) \leq D(1, y, z)$  and  $g(x) \leq D(1, y, z)$  then  $D(x, y, z) = (f(x) \cup g(x))$
- (ii) If  $f(x) \geq D(1, y, z)$  and  $g(x) \geq D(1, y, z)$  then  $D(x, y, z) \geq D(1, y, z)$

#### Proposition 3.2.9

Let  $L$  be a distributive lattice and  $D$  be a permuting tri- $(f, g)$ -derivation on  $L$  with the trace  $d$ , then

$$D(x \cap y) = (D(x) \cap f(y) \cup (g(x) \cap D(y))) \cup \{g(x) \cap f(y)\} \cap [D(x, x, y) \cup D(x, y, y)]$$

for every  $x, y \in L$

## RESULTS AND DISCUSSION

In this section, our main finding will be presented. The results of Balogun(2014) on principal derivations, Yazarli & Ozturk(2011) on permuting tri -  $f$ -derivations and Kacilioglu(2011) on permuting tri -  $(f, g)$  – derivations of lattices will be extended to derivation actions, permuting  $Z_4$  -  $f$  and  $Z_4$  -  $(f, g)$  – derivations of lattices.

### 4.1 Derivation actions

**Definition 2:** Let  $L$  be a lattice and  $X = L$ . Define a function  $D_g: L \rightarrow L$  by  $D_g(x) = gx \cap g$ , then  $D_g$  is a derivation such derivations are called derivation actions. For all  $x \in X$  and  $g \in L$ .

#### Proposition 4.1.1

Every derivation action of a lattice  $L$  is an isotone derivation.

#### Proof

Let  $D_g$  be a derivation action of a lattice  $L$ . Since for every  $x, y \in X$  and  $g \in L$ , we have

$$D_g(x) = gx \cap g \leq gy \cap g = D_g(y)$$

Hence

$D_g$  is isotone derivation.

#### Proposition 4.1.2

Let  $L$  be a lattice and  $D_g: L \rightarrow L$  be a derivation action, then the following conditions are equivalent

- (i)  $D_g$  is an isotone derivation
- (ii)  $D_g(x \cap y) = y \cap D_g(x)$

#### Proof

Suppose  $D_g$  is an isotone derivation actions, we have  $x \leq y$  implies that  $D_g(x) \leq D_g(y)$  for all  $x, y \in X$  and  $g \in L$

Therefore  $D_g$  is an isotone derivation actions.

(i)  $\rightarrow$  (ii)

$$D_g(x \cap y) = (D_g(x) \cap y) \cup (x \cap D_g(y)) \geq y \cap D_g(x) \quad (1)$$

Also  $x \cap y \leq x$  and  $x \cap y \leq y$

This implies that

$$D_g(x \cap y) \leq D_g(x) \quad \text{and} \quad D_g(x \cap y) \leq D_g(y)$$

Now

$$D_g(x \cap y) \leq D_g(x) \cap D_g(y) \leq y \cap D_g(x) \quad (2)$$

From equation (1) and (2), we get

$$D_g(x \cap y) = y \cap D_g(x)$$

(ii)  $\rightarrow$  (i)

Suppose  $D_g(x \cap y) = y \cap D_g(x)$  for all  $x, y \in X$  and  $g \in L$

Now

$$D_g(x \cap y) = (D_g(x) \cap y) \cup (x \cap D_g(y)) \geq y \cap D_g(x)$$

If  $x \leq y$ , since  $D_g(x \cap y) = y \cap D_g(x)$  and

$D_g(x \cap y) \leq D_g(y)$ , it implies that

$$D_g(y) = y \cap D_g(x),$$

Therefore, we have

$$D_g(y) \cup D_g(x) = (y \cap D_g(x)) \cup D_g(x) = D_g(x)$$

Hence

$$D_g(x) \leq D_g(y)$$

Proposition 4.1.3

Let  $L$  be a distributive lattice and  $D_1, D_2$  and  $D_3$  be isotone derivation actions on  $L$  defined by

$$((D_1 \cap D_2) \cup D_3)x = (D_1x \cap D_2x) \cup D_3x$$

$$((D_1 \cup D_2) \cap D_3)x = (D_1x \cup D_2x) \cap D_3x$$

Then  $(D_1 \cap D_2) \cup D_3$  and  $(D_1 \cup D_2) \cap D_3$  are also isotone derivations on  $L$

Proof

$$((D_1 \cap D_2) \cup D_3)(x \cap y) \leq ((D_1(x \cap y) \cap D_2(x \cap y)) \cup D_3(x \cap y))$$

$$\leq ((x \cap D_1y \cap x \cap D_2y) \cup x \cap D_3y)$$

$$\leq x \cap (D_1y \cap D_2y) \cup x \cap D_3y$$

$$\leq x \cap (D_1 \cap D_2) \cup D_3y \quad (1)$$

Similarly

$$((D_1 \cap D_2) \cup D_3)(y \cap x) \leq y \cap (D_1 \cap D_2) \cup D_3x \quad (2)$$

Combining equation (1) and (2) we have

$$((D_1 \cap D_2) \cup D_3)(x \cap y) = ((D_1 \cap D_2) \cup D_3)(x \cap y) \cup ((D_1 \cap D_2) \cup D_3)(y \cap x)$$

$$= ((D_1 \cap D_2) \cup D_3)(y \cap x)$$

Therefore

$$((D_1 \cap D_2) \cup D_3) \text{ is an isotone derivation}$$

action on  $L$

Also

$((D_1 \cap D_2) \cup D_3)$  is an isotone since

$$((D_1 \cap D_2) \cup D_3)(x \cup y) = ((D_1(x \cup y) \cap D_2(x \cup y)) \cup D_3(x \cup y))$$

$$= ((D_1x \cup D_1y \cap D_2x \cup D_2y) \cup (D_3x \cup D_3y))$$

$$= ((D_1x \cap D_2x) \cup (D_1y \cap D_2y)) \cup (D_3x \cup D_3y)$$

$$= (D_1 \cap D_2) \cup D_3x \cup ((D_1 \cap D_2) \cup D_3)y$$

$$= (D_1 \cap D_2) \cup D_3x \cup ((D_1 \cap D_2) \cup D_3)y$$

Similarly

$$(D_1 \cup D_2) \cap D_3 \text{ is an isotone}$$

derivation action on  $L$

4.2 Permuting  $Z_4$ - $f$ - and  $Z_4$  - ( $f, g$ ) - derivations of lattices

Example Let  $L = \{1, 2, 3, 4\}$ , We define the partial

ordering  $\leq$  on  $L$ , by the set

$\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$ .

The operation tables for  $\cup$  and  $\cap$  on  $L$  are

| $\cup$ | 1 | 2 | 3 | 4 |
|--------|---|---|---|---|
| 1      | 1 | 2 | 3 | 4 |
| 2      | 2 | 2 | 4 | 4 |
| 3      | 3 | 4 | 3 | 4 |
| 4      | 4 | 4 | 4 | 4 |

| $\cap$ | 1 | 2 | 3 | 4 |
|--------|---|---|---|---|
| 1      | 1 | 1 | 1 | 1 |
| 2      | 1 | 2 | 1 | 2 |
| 3      | 1 | 1 | 3 | 3 |
| 4      | 1 | 2 | 3 | 4 |

Since every pair of elements in  $L$  has both a join and a meet, then  $(L, \cup, \cap)$  is a lattice (under divides).

Note that  $1 \cup (2 \cap 3) = 1 \cup 1 = 1$  and  $(1 \cup 2) \cap$

$$(1 \cup 3) = 2 \cap 3 = 1$$

Therefore  $(1 \cup (2 \cap 3) = (1 \cup 2) \cap (1 \cup 3)$  for some

$1, 2, 3 \in L$ , Thus this lattice is distributive.

Definition 3: A mapping  $D: L \times L \times L \rightarrow L$  is called permuting  $Z_4$  if it satisfies the following conditions.

$$D(w, x, y, z) = D(w, x, z, y) = D(w, y, x, z)$$

$$= D(w, y, z, x) = D(w, z, y, x)$$

$$= D(w, z, x, y)$$

$$= D(x, w, y, z) = D(x, w, z, y) = D(x, y, w, z) =$$

$$D(x, y, z, w) = D(x, z, y, w) = D(x, z, w, y) =$$

$$= D(y, w, x, z) = D(y, w, z, x) = D(y, x, z, w) =$$

$$D(y, x, w, z) = D(y, z, w, x) = D(y, z, x, w) =$$

$$= D(z, w, x, y) = D(z, x, y, w) = D(z, x, w, y) =$$

$$D(z, w, y, x) = D(z, y, x, w) = D(z, y, w, z)$$

$D$  is a permuting  $Z_4$ -derivation if the following relations holds

$$(i) \quad D(v \cap w, x, y, z) = (D(w, x, y, z) \cap v) \cup (w \cap D(v, x, y, z))$$

$$(ii) \quad D(v, w, x, y \cap z) = (D(v, w, x, z) \cap y) \cup (z \cap D(v, w, x, y)) \quad \text{For all } v, w, x, y, z \in L$$

$Z_4$  is the group of permutation of the set  $\{1234\}$  and by cayley's theorem  $Z_4$  is isomorphic to a subgroup of  $S_4$  since the following groups  $(1,2)$ ,  $(2,3)$ ,  $(3,4)$  and  $(4,1)$  are normalizer of  $S_4$ .

**Definition 4:** Let  $L$  be a lattice ordered group. A permuting mapping  $D : L \times L \times L \times L \rightarrow L$  is called permuting  $Z_4$ - $f$ -derivation if there exists a function  $f : L \rightarrow L$  such that

$D(x \cap v, w, y, z) = D(x, w, y, z) \cap f(v) \cup f(x) \cap D(v, w, y, z)$  and it is called permuting  $Z_4$  - ( $f, g$ ) - derivation if there exists functions  $f, g : L \rightarrow L$  such that  $D(x \cap v, w, y, z) = D(x, w, y, z) \cap f(v) \cup (g(x) \cap D(v, w, y, z))$ , for all  $x, y, v, w, z \in L$

**Proposition 4.2.1**

For any  $x \in L$ ,  $D(x, x, x, x) = d(x) \leq f(x)$  where  $d(x) = D(x \cap x, x, x, x)$

**Proof**

$$\begin{aligned} D(x, x, x, x) &= D(x \cap x, x, x, x) = \\ &= (D(x, x, x, x) \cap f(x)) \cup (f(x) \cap D(x, x, x, x)) \\ &= D(x, x, x, x) \cap f(x) \\ &= D(x, x, x, x) \leq f(x) \quad \text{which} \\ &\quad \text{implies that} \\ &\quad d(x) \leq f(x) \end{aligned}$$

**Proposition 4.2.2**

Let  $L$  be a lattice ordered group and  $D$  be a permuting  $Z_4$ - $f$ -derivation on  $L$ , then for every  $x, y, w, z \in L$ , the following holds.

$$(i) \quad D(x, w, y, z) \leq f(x), \quad D(x, w, y, z) \leq f(w), \\ D(x, w, y, z) \leq f(y), \quad D(x, w, y, z) \leq f(z)$$

$$(ii) \quad D(x, y, z) \cap D(w, y, z) \leq D(x \cap w, y, z) \leq D(x, y, z) \cup D(w, y, z)$$

$$(iii) \quad D(x \cap w, y, z) \leq f(x) \cup f(w)$$

**Proof**

$$\begin{aligned} &\text{Since } x \cap x = x, \text{ then} \\ &D(x, w, y, z) = D(x \cap x, w, y, z) = D(x, w, y, z) \cap \\ &f(x) \cup f(x) \cap D(x, w, y, z) \end{aligned}$$

$$D(x, w, y, z) \cap f(x)$$

$$D(x, w, y, z) \leq f(x)$$

$$D(x, w, y, z) = D(w \cap w, x, y, z) = D(x, w, y, z) \cap f(w) \cup f(w) \cap D(x, w, y, z)$$

$$D(x, w, y, z) \cap f(w)$$

$$D(x, w, y, z) \leq f(w)$$

$$D(v, w, y, z) = D(v \cap v, w, y, z) = D(v, w, y, z) \cap f(v) \cup f(v) \cap D(v, w, y, z)$$

$$D(v, w, y, z) \cap f(v)$$

$$D(v, w, y, z) \leq f(v)$$

Similarly, we see that  $D(x, w, y, z) \leq f(y)$  and  $D(x, w, y, z) \leq f(z)$

(ii) Since  $D(x, w, y, z) \leq f(x)$  and  $D(v, w, y, z) \leq f(v)$  then from (i), we have

$$D(x, w, y, z) \cap D(v, w, y, z) = D(x \cap v, w, y, z) \leq D(x, w, y, z) \cup D(v, w, y, z)$$

$$\text{It implies that (a) } D(x, w, y, z) \cap D(v, w, y, z) = f(x) \cap D(v, w, y, z)$$

$$(b) \quad D(v, w, y, z) \cap D(x, w, y, z) = f(v) \cap D(x, w, y, z)$$

From (a) and (b) we get

$$\begin{aligned} D(x, w, y, z) \cap D(v, w, y, z) &\leq (f(x) \cap D(v, w, y, z)) \cup (f(v) \cap D(x, w, y, z)) \\ &= D(x \cap v, w, y, z) \end{aligned}$$

(iii) Furthermore, since

$$\begin{aligned} D(v, w, y, z) \cap f(x) &\leq D(v, w, y, z) \text{ and } f(v) \cap D(x, w, y, z) \leq D(x, w, y, z) \\ \text{then we get} \end{aligned}$$

$$f(x) \cap D(v, w, y, z) \cup (f(v) \cap D(x, w, y, z)) \leq D(v, w, y, z) \cup D(x, w, y, z) \text{ i.e}$$

$$D(x \cap v, w, y, z) \leq D(v, w, y, z) \cup D(x, w, y, z)$$

Since

$$D(v, w, y, z) \cap f(v) \leq f(v) \quad \text{and } D(x, w, y, z) \cap f(x) \leq f(x) \quad \text{then}$$

$$D(x \cap v, w, y, z) \leq f(x) \cup f(v)$$

**Proposition 4.2.3**

Let  $L$  be a lattice ordered group and  $D$  be a permuting  $Z_4$ - $f$ -derivation on  $L$ . If  $L$  is an increasing function, then  $v \leq x$  and  $D(x, w, y, z) = f(x)$  implies that  $D(v, w, y, z) \leq f(v)$  for every  $v, w, x, y, z \in L$

**Proof**

$$\begin{aligned} &\text{Suppose } v \leq x \text{ then } x \cap v = v, \text{ thus} \\ &D(v, w, y, z) = D(x \cap v, w, y, z) = \\ &D(x, w, y, z) \cap f(v) \cup f(x) \cap D(v, w, y, z) \end{aligned}$$

$$\begin{aligned} &= (f(x) \cap f(v)) \cup (f(x) \cap D(v, w, y, z)) = f(v) \cup \\ &(f(x) \cap D(v, w, y, z)) = f(v) \cup D(v, w, y, z) \end{aligned}$$

**Proposition 4.2.4**

Let  $D$  be a permuting  $Z_4$  -  $f$  - derivation on a distributive lattice  $L$ , then

$$D(x, w, y, z) \cap D(v, w, y, z) \leq D(x \cap v, w, y, z) \quad \text{for all } v, w, x, y, z \in L$$

**Proof**

$$\text{From proposition 4.2.1 } D(x, x, x, x) \leq f(x)$$

Since

$$D(x \cap v, w, y, z) = (D(v, w, y, z) \cap f(x)) \cup (f(v) \cap D(x, w, y, z)) \text{ then}$$

$$D(v, w, y, z) \cap f(x) \leq D(x \cap v, w, y, z) \text{ and}$$

$$D(x, w, y, z) \cap f(v) \leq D(x \cap v, w, y, z) \quad (1)$$

Also since

$$D(x \cap v, w, y, z) = D(v \cap x, w, y, z)$$

$$= (D(x, w, y, z) \cap f(v)) \cup (f(x) \cap D(v, w, y, z))$$

$$= D(x, w, y, z) \cap f(v) \leq D(x \cap v, w, y, z) \text{ and}$$

$$(v, w, y, z) \cap f(x) \leq D(x \cap v, w, y, z) \quad (2)$$

Combining (1) and (2) we get

$$(D(x, w, y, z) \cap f(v)) \cup (f(x) \cap D(v, w, y, z)) \leq D(x \cap v, w, y, z) \quad (3)$$

From equation (3), since  $L$  is a distributive lattice then we get

$$D(x, w, y, z) \cap D(v, w, y, z)$$

$$\leq f(v) \cup f(x) \cap D(v, w, y, z)$$

$$= (f(v) \cap D(v, w, y, z)) \cup (f(x) \cap D(v, w, y, z))$$

$$\leq D(x \cap v, w, y, z)$$

Proposition 4.2.5

Let  $L$  be a distributive lattice ordered group and  $D$  a permuting  $Z_4$ - $f$  - derivation on  $L$  where  $f(x \cup v) = f(x) \cup f(v)$ , then the following holds.

- (i) If  $D$  is isotone derivation, then  $D(x \cap v, w, y, z) = D(x, w, y, z) \cap D(v, w, y, z)$
- (ii)  $D$  is isotone derivation if and only if  $D(x \cup v, w, y, z) = D(x, w, y, z) \cup D(v, w, y, z)$

Proof

- (i) Since  $D$  is isotone derivation then  $D(x \cap v, w, y, z) \leq D(x, w, y, z) \cap D(v, w, y, z)$  from

Proposition 4.2.4 we have  $D(x, w, y, z) \cap D(v, w, y, z) = (D(x, w, y, z) \cap f(v)) \cap (f(x) \cap D(v, w, y, z)) = (D(x, w, y, z) \cap f(v)) \cap (f(x) \cap D(v, w, y, z)) \leq ((D(x, w, y, z) \cap f(v)) \cup (f(x) \cap D(v, w, y, z)))$

That

$$D(x \cap v, w, y, z) = D(x, w, y, z) \cap D(v, w, y, z)$$

- (ii) Let  $D$  be isotone derivation, then

$$D(v, w, y, z) = D(v, w, y, z) \cup (f(v) \cap D(x \cup v, w, y, z))$$

$$= (D(v, w, y, z) \cup f(v)) \cap (D(v, w, y, z) \cup D(x \cup v, w, y, z))$$

$$D(x \cup v, w, y, z) = f(v) \cap D(x \cup v, w, y, z)$$

Similarly  $D(x, w, y, z) = f(x) \cap D(x \cup v, w, y, z)$  therefore have

$$D(x, w, y, z) \cup D(v, w, y, z) = (f(v) \cap D(x \cup v, w, y, z)) \cup (f(x) \cap D(x \cup v, w, y, z))$$

=

$$(f(x) \cup f(v)) \cap D(x \cup v, w, y, z)$$

=

$$f(x \cup v) \cap D(x \cup v, w, y, z)$$

$$= D(x \cup v, w, y, z)$$

Conversely

$$\text{Let } D(x \cup v, w, y, z) = D(x, w, y, z) \cup D(v, w, y, z) \text{ and } x \leq v.$$

Since

$$D(v, w, y, z) = D(x \cup v, w, y, z) = D(x, w, y, z) \cup D(v, w, y, z), \text{ we get}$$

$$D(x, w, y, z) \leq D(v, w, y, z)$$

#### 4.3 Permuting $Z_4$ - ( $f, g$ ) - derivation of lattices

Definition 4: Let  $L$  be a lattice ordered group and  $D: L \times L \times L \times L \rightarrow L$  be a permuting mapping, then  $D$  is called permuting  $Z_4$  - ( $f, g$ )-derivation of  $L$  if there exists functions  $f, g: L \rightarrow L$  such that

$$D(x \cap v, w, y, z) = (D(x, w, y, z) \cap f(v)) \cup (g(x) \cap D(v, w, y, z)) \text{ for all } v, w, x, y, z \in L$$

Proposition 4.3.1

Let  $L$  be a lattice ordered group and  $D$  be the trace of permuting  $Z_4$ -( $f, g$ )- derivation on  $L$ , then

$$D(x) \leq (f(x) \cup g(x)) \text{ For all } x \in L$$

Proof

Since  $x \cap x = x$  for all  $x \in L$  and from the definition of trace, we have

$$d(x) = D(x, x, x, x) = D(x \cap x, x, x, x)$$

$$= D(x, x, x, x) \cap f(x) \cup g(x) \cap D(x, x, x, x)$$

Since  $D(x, x, x, x) \cap f(x) \leq f(x)$  and  $D(x, x, x, x) \cap g(x) \leq g(x)$  we get

$$d(x) \leq f(x) \cup g(x)$$

Proposition 4.3.2

Let  $L$  be a lattice ordered group and  $D$  a permuting  $Z_4$ -( $f, g$ )-derivation on  $L$ , then

- (i)  $D(x, w, y, z) \leq f(x) \cup g(x)$  (ii)  $D(x, w, y, z) \leq f(w) \cup g(w)$
- (iii)  $D(x, w, y, z) \leq f(y) \cup g(y)$  (iv)  $D(x, w, y, z) \leq f(z) \cup g(z)$

Proof

Since  $x \cap x = x$  for all  $x \in L$ , then we have

$$D(x, w, y, z) = D(x \cap x, w, y, z) \cap f(x) \cup g(x) \cap D(x, w, y, z)$$

Therefore  $D(x, w, y, z) \cap f(x) \leq f(x)$  and  $D(x, w, y, z) \cap g(x) \leq g(x)$  then

$$D(x, w, y, z) \leq f(x) \cup g(x)$$

Since  $w \cap w = w$  for all  $w \in L$ , then we have

$$D(x, w, y, z) = D(w \cap w, x, y, z) = (D(x, w, y, z) \cap f(w)) \cup (g(w) \cap D(x, w, y, z))$$

Since

$$D(x, w, y, z) \cap f(w) \leq f(w) \text{ and } D(x, w, y, z) \cap g(w) \leq g(w) \text{ then we get}$$

$D(x, w, y, z) \leq f(w) \cup g(w)$  Same holds for  $y, z$

### Proposition 4.3.3

Let  $D$  be a permuting  $Z_4$ -( $f, g$ )-derivation on  $L$ , If  $L$  has a least element 1, such that

$f(1) = 1$  and  $g(1) = 1$   
then  $D(1, w, y, z) = 1$  for all  $w, y, z \in L$

Proof

$D(1, w, y, z) = D(1 \cap 1, w, y, z) = D(1, w, y, z) \cap f(1) \cup g(1) \cap D(1 \cap 1, w, y, z)$

$$1 \cup 1 = 1$$

Hence  $D(1, w, y, z) = 1$

### Proposition 4.3.4

Let  $L$  be a lattice with a greatest element 4 and  $D$  a permuting  $Z_4$ -( $f, g$ )-derivation on  $L$

such that  $f(4) = g(4) = 4$ , then the following are valid

- (i) if  $f(x) \leq D(4, w, y, z)$  and  $g(x) \leq D(4, w, y, z)$  then  $D(x, w, y, z) = (f(x) \cup g(x)) \cap D(4, w, y, z)$
- (ii) If  $f(x) \geq D(4, w, y, z)$  and  $g(x) \geq D(4, w, y, z)$  then  $D(x, w, y, z) \geq D(4, w, y, z)$

Proof

$$\begin{aligned} (i) D(4, w, y, z) &= D(4 \cap 4, w, y, z) = (D(4, w, y, z) \cap f(4)) \cup (g(4) \cap D(4, w, y, z)) \\ &= D(4, w, y, z) \cap (f(4) \cup g(4)) \\ &= \end{aligned}$$

$$D(4, w, y, z) \leq (f(4) \cup g(4))$$

When replacing 4 with  $x$ , then we get

$$D(x, w, y, z) \leq (f(x) \cup g(x))$$

Similarly to

$$D(x, w, y, z) \geq D(4, w, y, z) \text{ for all } w, x, y, z \in L$$

### Proposition 4.3.5

Let  $D$  be a permuting  $Z_4$  - ( $f, g$ ) - derivation on a distributive lattice  $L$ , then

$$D(x, w, y, z) \cap D(v, w, y, z) \leq D(x \cap v, w, y, z) \text{ for all } v, w, x, y, z \in L$$

Proof

From proposition 4.3.1,  $D(x, x, x, x) \leq f(x) \cup g(x)$

Since

$$D(x \cap v, w, y, z) = (D(v, w, y, z) \cap f(x)) \cup (g(v) \cap D(x, w, y, z)) \text{ then}$$

$$D(v, w, y, z) \cap f(x) \leq D(x \cap v, w, y, z) \text{ and}$$

$$D(x, w, y, z) \cap g(v) \leq D(x \cap v, w, y, z) \quad (1)$$

Also since

$$\begin{aligned} D(x \cap v, w, y, z) &= D(v \cap x, w, y, z) \\ &= (D(x, w, y, z) \cap f(v)) \cup (g(x) \cap D(v, w, y, z)) \end{aligned}$$

$$\begin{aligned} &= D(x, w, y, z) \cap f(v) \leq D(x \cap v, w, y, z) \text{ and} \\ &(v, w, y, z) \cap g(x) \leq D(x \cap v, w, y, z) \quad (2) \end{aligned}$$

Combining (1) and (2) we get

$$(D(x, w, y, z) \cap f(v)) \cup (g(x) \cap D(v, w, y, z)) \leq D(x \cap v, w, y, z) \quad (3)$$

From equation (3), since  $L$  is a distributive lattice then we get

$$\begin{aligned} &D(x, w, y, z) \cap D(v, w, y, z) \\ &\leq f(v) \cup g(x) \cap D(v, w, y, z) \\ &= (f(v) \cap D(v, w, y, z)) \cup (g(x) \cap D(v, w, y, z)) \\ &\leq D(x \cap v, w, y, z) \end{aligned}$$

## CONCLUSION

In this paper, we used the notions and properties of derivations, permuting tri- $f$ -derivations and permuting tri- $(f, g)$ - derivations on lattices and established derivation actions, permuting  $Z_4$  - $f$  and  $Z_4$ -( $f, g$ )-derivations of lattices. The study explored the behavior of these derivations, including their trace, modular, isotone and how they interact with lattice. For further research, permuting  $Z_4$  can be extended to automorphism derivation on lattices, for instance, if  $L$  is a lattice ordered group and  $g \in L$ , we can define  $D_L: L \rightarrow L$  by  $D_L(x) = g^{-1}xg$ , for all  $x \in L$ , then we can show that  $D_L$  is an automorphism derivation of  $L$ .

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