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Robust Calibrated Optional Randomized Response Techniques for the Estimation of Sensitive Variable Information with Applications to SITs Data

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ABSTRACT
The use of existing RRTs Models in STIs research offers several advantages, such as increased response accuracy, reduced social desirability bias, and enhanced confidentiality protection. However, the existing RRT models utilize no auxiliary information which can enhance the accuracy and precision of estimate for prevalence of STIs. In addition, the estimators of the existing RRT models are prone to outliers or extreme values being them sample means of the interest variables which in turn can reduce the efficiency and accuracy levels of models. In this research, we proposed new classes of randomized response technique called calibrated three optional randomized response techniques. These models were created by adjusting current RRT models using calibration techniques. The aim was to enhance C-RRT models to be more efficient, stable, and robust compared to existing alternatives. The research established theoretical properties, including estimators, variances, privacy levels, and a composite metric for efficiency and privacy, to evaluate the robustness and applicability of the proposed models. Empirical studies were conducted using simulated data to support the theoretical findings, and demonstrating that the R-CTHORRT models exhibited lower variances, higher relative efficiency, enhanced privacy levels, and a better combined metric of variance and privacy. This indicates the superiority of the C-THORRT models over existing RRT models.

Keywords:
Respondent,
Sensitive variable,
Privacy,
Calibration,
Auxiliary information.

INTRODUCTION

One of the important areas of sampling survey is estimation of population parameter, especially estimation of sensitive characteristics like cases of prevalence of STIs, sexual harassment, cases of rape, illegal drug use etc., whose information cannot be obtained from respondents with higher probability of truth when direct method of data collection is employed. The respondents fear that if they provide the real value of the sensitive variable, they will be stigmatized or punished by the law. For example, many rape victims won't speak out, out fear of being stigmatized or humiliated or disgraced and many will not go to the hospital to explain what happened to them, hence, the society is not so safe in this situation. Obtaining information from these respondents through direct method is not ideal, hence, we apply the randomized response technique (RRT). To obtain more reliable information from the respondents, Warner (1965) came up with an ingenious approach for data collection known as the randomized response technique (RRT). This method guarantees the privacy of the respondents and also conceals their responses. The technique developed in 1965 was specifically designed for qualitative variable. In 1971, Warner introduced randomized response technique (RRT) model for quantitative variable. In 1976, Pollock and Bek, reintroduced this technique as additive model. Randomized response techniques (RRT) are valuable techniques used in surveys or questionnaires to collect sensitive data while maintaining the privacy of participants. This is especially beneficial when researchers are addressing controversial or sensitive subjects. By ensuring confidentiality, RRT can boost participation rates in surveys or studies related to delicate topics. Respondents are more inclined to take part if they believe their privacy is protected. Various authors have suggested different RRT models and estimators for estimating population parameters of sensitive qualitative and quantitative variables. These authors include Eichhorn and Hayre (1983), Gupta et al. (2002),

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Bar-Lev at al. (2004), Singh and Mathur (2005), Gupta et al. (2006), Gupta and Shabbir (2007), Saha (2007), Gjestvang and Singh (2009), Huang (2010), Gupta et al. (2010), Diana and Perri (2010), Diana and Perri (2011), Hussain (2012), Mehta et al. (2012), Singh and Tarray (2016), Tarray and Singh (2017), Gupta et al. (2018), Saleem et al. (2019), Azeem and Salam (2023), Azeem et al. (2023), Yunusa et al. (2025a,b).

The estimates produced by the estimation techniques in sampling survey are often being enhanced with the use of auxiliary information. Auxiliary information means additional data that is related to the variable of interest and can be used to improve estimators' performance in sample surveys. Auxiliary information can help reduce the variance of estimators, leading to more precise estimates (Cochran, 1940). This is particularly useful when dealing with small sample sizes or when estimating small subpopulations. Auxiliary information can be used to adjust for potential biases in the sample, such as non-response bias or under-coverage bias, resulting in more accurate estimates. By incorporating auxiliary information, estimators can become more efficient, requiring smaller sample sizes to achieve the same level of precision (Cochran, 1942). This can help lower the costs linked to data collection. Auxiliary information can be used to address missing data in surveys, either through imputation or by guiding the selection of weights in estimation methods. This information can be sourced from various places, such as administrative records, census data, or prior surveys (Singh, 2003). By utilizing existing data, researchers can enhance estimators and enrich their analyses. Incorporating auxiliary information to align estimates with known population benchmarks or external data sources can lead to improved comparisons across different surveys or time periods. One method for integrating auxiliary information into models or estimators is through the use of calibration techniques (Audu et al., 2024a,b,c, Audu et al. 2025b).

Calibration of estimators in sample surveys is a technique used to correct differences between the survey sample and the target population, especially when the sample isn't fully representative. This process adjusts estimators to match known population totals or benchmarks, thereby reducing bias and enhancing estimate accuracy. Calibration mitigates the effects of sampling variability that can occur when a sample does not fully reflect the target population. It can also utilize auxiliary information from other sources, such as administrative records or census data, to further refine and improve the estimates. Jabeen et al. (2024) developed calibrated estimators for sensitive variables under stratified random sampling. Also, recently, Audu et al. (2025a) developed calibrated models and estimators with two groups of respondents for sensitive variables under simple random sampling. In the present, we aimed at proposing randomized response calibrated models and estimators for estimation of mean of sensitive study variable for three groups of respondents under simple random sampling schemes.

Randomized Response Techniques (RRTs) have been developed as a valuable tool for collecting sensitive information in various research areas, including the study of sexually transmitted infections (STIs). These techniques aim to protect respondent confidentiality and reduce social desirability bias, which can lead to underreporting of sensitive behaviors or conditions. RRTs have been applied in STI research to estimate the prevalence of STIs, such as HIV/AIDS, syphilis, and others, as well as related risk behaviors (see Baltagi & Wannous, 2020; Park, Park & Kim, 2020). These techniques have been particularly useful for collecting data from hard-to-reach populations or those who may be less likely to disclose sensitive information due to social stigma or discrimination. Examples of RRT applications in STI research include surveys of sex workers, intravenous drug users, and men who have sex with men. The continued development and refinement of RRTs for use in STIs research are essential to improve data collection, inform public health interventions, and reduce the burden of STIs. However, future directions include the development of robust RRT models, integration with other data collection methods, and adaptation to digital platforms. Additionally, more research is needed to evaluate the effectiveness and scalability of RRTs in diverse settings and populations. In conclusion, randomized response techniques have significantly contributed to the study of STIs by providing a valuable tool to collect sensitive data while protecting respondent confidentiality. Continued advancements in this field will further enhance the accuracy and reliability of data collected on STIs, leading to more effective public health interventions.

Leveraging on the approach of calibration, this study aimed at modifying RRT model of Azeem et al. (2024) to propose robust and efficient calibrated randomized response technique (C-RRT) models to obtain estimates with less biases and higher precisions for estimating the rate of prevalence of STIs.**pro**

Let Y be a sensitive study variable which correlated with an auxiliary variable X from a population consists of N elements from which a sample of size n elements is drawn. Let S and T be scrambling variables which are uncorrelated with Y whose mean and variances are assumed to be known. Let Z be the scrambling response of Y. Then, the following notations are defined.

$E(Y_i)=\mu_y$: population mean of Y, $Var(Y_i)=\sigma_y^2$: population variance of Y, $E(S)=\mu_s=\theta$: population mean of S, $Var(S)=\sigma_s^2$: population variance of S, $E(T)=\mu_T$:population mean of T, $Var(T)=\sigma_T^2$: population variance of T, $\bar{X}=N^{-1}\sum_{i=1}^{N}X_i$: Population mean of X, $\sigma_x^2=N^{-1}\sum_{i=1}^{N}\left(x_i-\bar{X}\right)^2$: population variance of , $\bar{x}=n^{-1}\sum_{i=1}^{n}x_i$: sample mean of X (Audu et al., 2025a).

Azeem et al. (2024) proposed RRT model with one scramble variable as in (2.1).

$$Z_{(AZ)i}=\begin{cases} Y & \text{with probability } 1-W \\ Y+S-J & \text{with probability } WA \\ TY+SJ & \text{with probability } W(1-A) \end{cases} \quad 2.1$$

where W denote the sensitivity level and A denotes a constant such that $0<A<1$.

The estimator of population means μ_y and its variance using Gupta et al. (2022) model under the assumption that $E(S)=\mu_s, E(T)=1, E(J)=\mu_J$ are given as in (2.2) and (2.3) respectively and the combined metric of privacy level and efficiency denoted by δ_G is given as in (2.4).

$$\hat{\mu}_{AZ_1}=\frac{1}{n}\sum_{i=1}^{n}Z_i+WA\mu_j, \quad 2.2$$

$$Var\left(\hat{\mu}_{AZ_1}\right)=\frac{1}{n}\begin{pmatrix}\sigma_y^2+WA\left(\sigma_s^2+\sigma_j^2+\mu_j^2-2\mu_j\mu_y\right)+ \\ W(1-A)\begin{Bmatrix}\sigma_T^2\left(\sigma_y^2+\mu_y^2\right) \\ +\sigma_s^2\left(\sigma_j^2+\mu_j^2\right)\end{Bmatrix}\end{pmatrix} \quad 2.3$$

$$\delta_{AZ_1}=\frac{\begin{pmatrix}\sigma_y^2+WA\left(\sigma_s^2+\sigma_j^2+\mu_j^2-2\mu_j\mu_y\right)+ \\ W(1-A)\begin{Bmatrix}\sigma_T^2\left(\sigma_y^2+\mu_y^2\right) \\ +\sigma_s^2\left(\sigma_j^2+\mu_j^2\right)\end{Bmatrix}\end{pmatrix}}{n\begin{pmatrix}WA\left(\sigma_s^2+\sigma_j^2+\mu_j^2\right)+W(1-A) \\ \left\{\sigma_T^2\left(\sigma_y^2+\mu_y^2\right)+\sigma_s^2\left(\sigma_j^2+\mu_j^2\right)\right\}\end{pmatrix}}. \quad 2.4$$

MATERIALS AND METHODS

Proposed Calibration Estimators

Consider G_1, G_2 and G_3 as the sets of respondents belonging to the first second and third categories of Z respectively in all the RRT models stated in (2.1) having elements n_1, n_2 and n_3. Then, the models and their estimators can be generally be written as in (3.1) and (3.2) respectively.

$$Z_{.i}^{(b)}=\begin{cases}\theta(y)_{1i} & \text{with probability} \quad p_1^*, \quad i\in G_1 \\ \theta(y)_{2i} & \text{with probability} \quad p_2^*, \quad i\in G_2 \\ \theta(y)_{3i} & \text{with probability } 1-p_1^*-p_2^*, \quad i\in G_3\end{cases} \quad (3.1)$$

$$\hat{\mu}_y^{(b)}=\sum_{i\in G_1}\varpi_1\theta(y)_{1i}+\sum_{i\in G_2}\varpi_2\theta(y)_{2i}+\sum_{i\in G_3}\varpi_3\theta(y)_{3i} \quad 3.2$$

where $\varpi_1=\varpi_2=\varpi_3=\frac{1}{n},\ \theta(y)_{1i}=Y_i$,

$\theta(y)_{2i}=Y_i+S-J$, $\theta(y)_{3i}=TY_i+SJ$,

$p_1^*=1-W$, $p_2^*=WA$, $1-p_1^*-p_2^*=W(1-A)$.

Motivated by Audu *et al.* (2024a,b,c), this study proposed two (2) classes of calibration Schemes and Estimators to obtain two new classes of RRT models for estimating sensitive variables.

First Proposed calibration estimator

The first proposed estimator and calibration scheme are defined as in (3.3) and (3.4) respectively

$$\hat{\mu}_{\bullet\bullet}^{(1j)}=\sum_{i\in G_1}\varpi_{11i}\theta(y)_{1i}+\sum_{i\in G_2}\varpi_{12i}\theta(y)_{2i}+\sum_{i\in G_3}\varpi_{13i}\theta(y)_{3i} \quad 3.3$$

where W_{11i} and W_{12i} are the new calibration weights to be obtained by minimizing the chi-square distance Z defined as

$$\left.\begin{array}{l}\min \psi_1= \\ \displaystyle\sum_{i\in G_1}\frac{(n\varpi_{11i}-1)^2}{2n\phi_{11i}}+\sum_{i\in G_2}\frac{(n\varpi_{12i}-1)^2}{2n\phi_{12i}}+\sum_{i\in G_3}\frac{(n\varpi_{13i}-1)^2}{2n\phi_{13i}} \\ s.t. \quad \displaystyle\sum_{i\in G_1}\varpi_{11i}x_{1i}+\sum_{i\in G_2}\varpi_{12i}x_{2i}+\sum_{i\in G_3}\varpi_{13i}x_{3i}=\mu_x\end{array}\right\} \quad 3.4$$

To compute ϖ_{11i}, ϖ_{12i} and ϖ_{13i} in (3.9), Lagrange function L_3 is defined as in (3.5).

$$L_3 = \sum_{i\in G_1}\frac{(n\varpi_{11i}-1)^2}{2n\phi_{11i}}+\sum_{i\in G_2}\frac{(n\varpi_{12i}-1)^2}{2n\phi_{12i}}+\sum_{i\in G_3}\frac{(n\varpi_{13i}-1)^2}{2n\phi_{13i}} -\varphi\left(\sum_{i\in G_1}\varpi_{11i}x_{1i}+\sum_{i\in G_2}\varpi_{12i}x_{2i}+\sum_{i\in G_2}\varpi_{13i}x_{2i}-\mu_x\right) \quad 3.5$$

Partially differentiating (3.5) with respect to ϖ_{11i} , ϖ_{12i} , ϖ_{13i} and φ , equate the results to zero, we obtained (3.6)-(3.9)

$$\varpi_{11i}=n^{-1}\left(1+\varphi\phi_{11i}x_{1i}\right) \quad 3.6$$

$$\varpi_{12i}=n^{-1}\left(1+\varphi\phi_{12i}x_{2i}\right) \quad 3.7$$

$$\varpi_{13i}=n^{-1}\left(1+\varphi\phi_{13i}x_{3i}\right) \quad 3.8$$

$$\sum_{i\in G_1}\varpi_{11i}x_{1i}+\sum_{i\in G_2}\varpi_{12i}x_{2i}+\sum_{i\in G_3}\varpi_{12i}x_{2i}=\mu_x \quad 3.9$$

By substituting (3.6) (3.7) and (3.8) into (3.9) and solve for λ , (3.10) is obtained.

$$\varphi = \left(n\mu_x-\left(\sum_{i\in G_1}x_{1i}+\sum_{i\in G_2}x_{2i}+\sum_{i\in G_3}x_{3i}\right)\right)\left(\sum_{i\in G_1}\phi_{11i}x_{1i}^2+\sum_{i\in G_2}\phi_{12i}x_{2i}^2+\sum_{i\in G_2}\phi_{13i}x_{3i}^2\right)^{-1} \quad 3.10$$

Substituting the value of φ into (3.6) (3.7) and (3.8), the expressions for ϖ_{11i} , ϖ_{12i} , and ϖ_{13i} are obtained as in (3.11) (3.12) and (3.13) respectively.

$$\varpi_{11i} = \frac{1}{n}+\phi_{11i}x_{1i}\left(\mu_x-n^{-1}\left(\sum_{i\in G_1}x_{1i}+\sum_{i\in G_2}x_{2i}+\sum_{i\in G_3}x_{3i}\right)\right)\left(\sum_{i\in G_1}\phi_{11i}x_{1i}^2+\sum_{i\in G_2}\phi_{12i}x_{2i}^2+\sum_{i\in G_3}\phi_{13i}x_{3i}^2\right)^{-1} \quad 3.11$$

$$\varpi_{12i} = \frac{1}{n}+\phi_{12i}x_{2i}\left(\mu_x-n^{-1}\left(\sum_{i\in G_1}x_{1i}+\sum_{i\in G_2}x_{2i}+\sum_{i\in G_3}x_{3i}\right)\right)\left(\sum_{i\in G_1}\phi_{11i}x_{1i}^2+\sum_{i\in G_2}\phi_{12i}x_{2i}^2+\sum_{i\in G_3}\phi_{13i}x_{3i}^2\right)^{-1} \quad 3.12$$

$$\varpi_{13i} = \frac{1}{n}+\phi_{13i}x_{3i}\left(\mu_x-n^{-1}\left(\sum_{i\in G_1}x_{1i}+\sum_{i\in G_2}x_{2i}+\sum_{i\in G_3}x_{3i}\right)\right)\left(\sum_{i\in G_1}\phi_{11i}x_{1i}^2+\sum_{i\in G_2}\phi_{12i}x_{2i}^2+\sum_{i\in G_3}\phi_{13i}x_{3i}^2\right)^{-1} \quad 3.13$$

Substituting (3.11) (3.12) and (3.13) in (3.3) to obtain the proposed calibration estimator as in (3.14)

$$\hat{\mu}_{gg}^{(1j)}=\hat{\mu}_{gg}+\hat{\beta}_{ij}^{*}\left(\mu_x-\bar{x}\right) \quad 3.14$$

where

$$\bar{x} = n^{-1}\left(\sum_{i\in G_1}x_{1i}+\sum_{i\in G_2}x_{2i}+\sum_{i\in G_3}x_{3i}\right),$$

$$\hat{\beta}_{ij}^{*}=\frac{\sum_{i\in G_1}\phi_{11i}x_{1i}\theta(y)_{1i}+\sum_{i\in G_2}\phi_{12i}x_{2i}\theta(y)_{2i}+\sum_{i\in G_3}\phi_{13i}x_{3i}\theta(y)_{3i}}{\sum_{i\in G_1}\phi_{11i}x_{1i}^2+\sum_{i\in G_2}\phi_{12i}x_{2i}^2+\sum_{i\in G3}\phi_{13i}x_{3i}^2}$$

.

Case 1: Setting $\phi_{11i}=\phi_{12i}=\phi_{13i}=1$ in (3.14), member of $\hat{\mu}_{gg}^{(1j)}$ denoted by $\hat{\mu}_{gg}^{(11)}$ is obtained as in (3.15)

$$\hat{\mu}_{gg}^{(11)}=\hat{\mu}_{gg}+\hat{\beta}_{11}^{*}\left(\mu_x-\bar{x}\right) \quad 3.15$$

where

$$\hat{\beta}_{11}^{*}=\frac{\sum_{i\in G_1}x_{1i}\theta(y)_{1i}+\sum_{i\in G_2}x_{2i}\theta(y)_{2i}+\sum_{i\in G_3}x_{3i}\theta(y)_{3i}}{\sum_{i\in G_1}x_{1i}^2+\sum_{i\in G_2}x_{2i}^2+\sum_{i\in G3}x_{3i}^2}$$

Case 2: Setting $\phi_{11i}=x_{1i}^{-1}$, $\phi_{12i}=x_{2i}^{-1}$, $\phi_{13i}=x_{3i}^{-1}$ in (3.14), member of $\hat{\mu}_{gg}^{(1j)}$ denoted by $\hat{\mu}_{gg}^{(12)}$ is obtained as in (3.16)

$$\hat{\mu}_{gg}^{(12)}=\hat{\mu}_{gg}+\hat{\beta}_{12}^{*}\left(\mu_x-\bar{x}\right) \quad 3.16$$

where

$$\hat{\beta}_{12}^{*}=\frac{\sum_{i\in G_1}\theta(y)_{1i}+\sum_{i\in G_2}\theta(y)_{2i}+\sum_{i\in G_2}\theta(y)_{2i}}{\sum_{i\in G_1}x_{1i}+\sum_{i\in G_2}x_{2i}+\sum_{i\in G_3}x_{3i}}=\frac{\bar{z}^{*}}{\bar{x}}.$$

The resultant estimator $\hat{\mu}_{gg}^{(1j)}$ obtained in (3.14) can be expressed as in (3.17)

$$\hat{\mu}_{\bullet\bullet}^{(1j)} = \sum_{i\in G_1}\varpi_1\left\{\theta(y)_{1i}+\hat{\beta}_{1j}^{*}\left(\mu_x-x_{1i}\right)\right\}+\sum_{i\in G_2}\varpi_2\left\{\theta(y)_{2i}+\hat{\beta}_{1j}^{*}\left(\mu_x-x_{2i}\right)\right\}+\sum_{i\in G_3}\varpi_3\left\{\theta(y)_{3i}+\hat{\beta}_{1j}^{*}\left(\mu_x-x_{3i}\right)\right\} \quad 3.17$$

Compared (3.17) with (3.2), the first proposed modified RRT model is obtained as in (3.18).

$$Z_{gg}^{(1j)} = \begin{cases}\theta(y)_1+\hat{\beta}_{1j}^{*}\left(\mu_x-X_1\right) & with\ prob.\ p_1^{*}\\ \theta(y)_2+\hat{\beta}_{1j}^{*}\left(\mu_x-X_2\right) & with\ prob.\ p_2^{*}\\ \theta(y)_3+\hat{\beta}_{1j}^{*}\left(\mu_x-X_3\right) & with\ prob.\ 1-p_1^{*}-p_2^{*}\end{cases} \quad 3.18$$

Second Proposed calibration estimator

The second proposed estimator and calibration scheme are defined as in (3.19) and (3.20) respectively.

$$\hat{\mu}_{\bullet\bullet}^{(2j)} = \sum\nolimits_{i\in G_1}\varpi_{21i}\theta(y)_{1i}+\sum\nolimits_{i\in G_2}\varpi_{22i}\theta(y)_{2i}+\sum\nolimits_{i\in G_3}\varpi_{23i}\theta(y)_{3i} \quad 3.19$$

where ϖ_{21i}, ϖ_{22i} and ϖ_{23i} are the new calibration weights to be obtained by minimizing the chi-square distance ψ_2 defined as

$$\left.\begin{array}{l}\min\ \psi_2 = \\ \sum_{i\in G_1}\frac{\left(n\varpi_{21i}-1\right)^2}{2n\phi_{21i}}+\sum_{i\in G_2}\frac{\left(n\varpi_{22i}-1\right)^2}{2n\phi_{22i}}+\sum_{i\in G_3}\frac{\left(n\varpi_{23i}-1\right)^2}{2n\phi_{23i}}\\ s.t.\quad \sum_{i\in G_1}\varpi_{21i}x_{1i}+\sum_{i\in G_2}\varpi_{22i}x_{2i}+\sum_{i\in G_3}\varpi_{23i}x_{3i}=\mu_x\\ \sum_{i\in G_1}\varpi_{21i}+\sum_{i\in G_2}\varpi_{22i}+\sum_{i\in G_3}\varpi_{23i}=\sum_{i\in G_1}\varpi_1+\sum_{i\in G_2}\varpi_2+\sum_{i\in G_3}\varpi_3\end{array}\right\} \quad 3.20$$

To compute ϖ_{21i}, ϖ_{22i} and ϖ_{23i}, Lagrange multiplier function L_2 is defined as in (3.21)

$$L_4 = \sum_{i\in G_1}\frac{\left(n\varpi_{21i}-1\right)^2}{2n\phi_{21i}}+\sum_{i\in G_2}\frac{\left(n\varpi_{22i}-1\right)^2}{2n\phi_{22i}}+\sum_{i\in G_3}\frac{\left(n\varpi_{23i}-1\right)^2}{2n\phi_{23i}}-\varphi_1\left(\begin{array}{l}\sum_{i\in G_1}\varpi_{21i}x_{1i}+\sum_{i\in G_2}\varpi_{22i}x_{2i}\\ +\sum_{i\in G_3}\varpi_{23i}x_{3i}-\mu_x\end{array}\right)-\varphi_2\left(\begin{array}{l}\sum_{i\in G_1}\varpi_{21i}+\sum_{i\in G_2}\varpi_{22i}+\sum_{i\in G_3}\varpi_{23i}-\\ \sum_{i\in G_1}\varpi_1-\sum_{i\in G_2}\varpi_2-\sum_{i\in G_3}\varpi_3\end{array}\right) \quad 3.21$$

Partially differentiating (3.21) with respect to ϖ_{21i}, ϖ_{22i}, ϖ_{23i}, φ_1 and φ_2, equate the results to zero,, (3.22), (3.23), (3.24), (3.25) and (3.26) are obtained respectively

$$\varpi_{21i} = n^{-1}\left(1+\varphi_1\phi_{21i}x_{1i}+\varphi_2\phi_{21i}\right) \quad 3.22$$

$$\varpi_{22i} = n^{-1}\left(1+\varphi_1\phi_{22i}x_{1i}+\varphi_2\phi_{22i}\right) \quad 3.23$$

$$\varpi_{23i} = n^{-1}\left(1+\varphi_1\phi_{23i}x_{1i}+\varphi_2\phi_{23i}\right) \quad 3.24$$

$$\sum_{i\in G_1}\varpi_{21i}x_{1i}+\sum_{i\in G_2}\varpi_{22i}x_{2i}+\sum_{i\in G_3}\varpi_{23i}x_{2i}=\mu_x \quad 3.25$$

$$\sum_{i\in G_1}\varpi_{21i}+\sum_{i\in G_2}\varpi_{22i}+\sum_{i\in G_3}\varpi_{23i} = \sum_{i\in G_1}\varpi_1+\sum_{i\in G_2}\varpi_2+\sum_{i\in G_3}\varpi_3 \quad 3.26$$

By substituting (3.22), (3.23) and (3.24) into (3.25) and (3.26), equation (3.27) is obtained

$$\begin{pmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\begin{pmatrix}\tau_4\\ 0\end{pmatrix} \quad 3.27$$

where $\tau_1 = \frac{1}{n}\left(\sum_{i\in G_1}\phi_{21i}x_{1i}^2+\sum_{i\in G_2}\phi_{22i}x_{2i}^2+\sum_{i\in G_3}\phi_{22i}x_{3i}^2\right)$,

$$\tau_2 = \frac{1}{n}\left(\sum_{i\in G_1}\phi_{21i}x_{1i}+\sum_{i\in G_2}\phi_{22i}x_{2i}+\sum_{i\in G_2}\phi_{22i}x_{3i}\right),$$

$$\tau_3 = \frac{1}{n}\left(\sum_{i\in G_1}\phi_{21i}+\sum_{i\in G_2}\phi_{22i}+\sum_{i\in G_3}\phi_{23i}\right),$$

$$\tau_4 = \mu_x-\frac{1}{n}\left(\sum_{i\in G_1}x_{1i}+\sum_{i\in G_2}x_{2i}+\sum_{i\in G_3}x_{3i}\right).$$

By solving (3.24) for φ_1 and φ_2, (3.28) is obtained.

$$\varphi_1 = \frac{\tau_3\tau_4}{\tau_1\tau_3-\tau_2^2},\ \varphi_2 = \frac{-\tau_2\tau_4}{\tau_1\tau_3-\tau_2^2} \quad 3.28$$

Substituting the values of φ_1 and φ_2 into (3.22) (3.23) and (3.24), the expressions for ϖ_{21i} ϖ_{22i} and ϖ_{23i} are obtained as in (3.29), (3.30) and (3.31) respectively.

$$\varpi_{21i} = n^{-1}\left(1+\frac{\tau_3\tau_4}{\tau_1\tau_3-\tau_2^2}\phi_{21i}x_{1i}-\frac{\tau_2\tau_4}{\tau_1\tau_3-\tau_2^2}\phi_{21i}\right) \quad 3.29$$

$$\varpi_{22i} = n^{-1}\left(1+\frac{\tau_3\tau_4}{\tau_1\tau_3-\tau_2^2}\phi_{22i}x_{2i}-\frac{\tau_2\tau_4}{\tau_1\tau_3-\tau_2^2}\phi_{22i}\right) \quad 3.30$$

$$\varpi_{23i} = n^{-1}\left(1+\frac{\tau_3\tau_4}{\tau_1\tau_3-\tau_2^2}\phi_{23i}x_{3i}-\frac{\tau_2\tau_4}{\tau_1\tau_3-\tau_2^2}\phi_{23i}\right) \quad 3.31$$

Substituting (3.29), (3.30) and (3.31) in (3.32), the proposed calibration estimator is obtained as in (3.32).

$$\hat{\mu}_{gg}^{(2j)} = \hat{\mu}_{gg} + \hat{\beta}_{2j}^{*}\left(\mu_x - \bar{x}\right) \quad 3.32$$

where $\hat{\beta}_{2j}^{*} = \frac{\tau_3\tau_5-\tau_2\tau_6}{\tau_1\tau_3-\tau_2^2}$,

$$\tau_5 = \frac{1}{n}\left(\sum_{i\in G_1}\phi_{21i}x_{1i}\theta(y)_{1i} + \sum_{i\in G_2}\phi_{22i}x_{2i}\theta(y)_{2i} + \sum_{i\in G_3}\phi_{23i}x_{3i}\theta(y)_{3i}\right),$$

$$\tau_6 = \frac{1}{n}\left(\sum_{i\in G_1}\phi_{21i}\theta(y)_{1i} + \sum_{i\in G_2}\phi_{22i}\theta(y)_{2i} + \sum_{i\in G3}\phi_{23i}\theta(y)_{3i}\right).$$

Case 1: Setting $\phi_{21i} = \phi_{22i} = 1$ in (3.32), member of $\hat{\mu}_{gg}^{(2j)}$ denoted by $\hat{\mu}_{gg}^{(21)}$ is obtained as in (3.33)

$$\hat{\mu}_{gg}^{(21)} = \hat{\mu}_{gg} + \hat{\beta}_{21}^{*}\left(\mu_x - \bar{x}\right) \quad 3.33$$

where $\hat{\beta}_{21}^{*} = \dfrac{\sum_{i\in G_1}x_{1i}\theta(y)_{1i} + \sum_{i\in G_2}x_{2i}\theta(y)_{2i} + \sum_{i\in G_3}x_{3i}\theta(y)_{3i} - \bar{x}\left(\sum_{i\in G_1}\theta(y)_{1i} + \sum_{i\in G_2}\theta(y)_{2i} + \sum_{i\in G_3}\theta(y)_{3i}\right)}{\sum_{i\in G_1}x_{1i}^2 + \sum_{i\in G_2}x_{2i}^2 + \sum_{i\in G_3}x_{3i}^2 - n\bar{x}^2}$

Case2: Setting $\phi_{21i} = x_{1i}^{-1}$, $\phi_{22i} = x_{2i}^{-1}, \phi_{23i} = x_{3i}^{-1}$ in (3.32), member of $\hat{\mu}_{gg}^{(2j)}$ denoted by $\hat{\mu}_{gg}^{(22)}$ is obtained as in (3.34).

$$\hat{\mu}_{g}^{(22)} = \hat{\mu}_{g} + \hat{\beta}_{22}\left(\mu_x - \bar{x}\right) \quad 3.34$$

where $\hat{\beta}_{22}^{*} = \dfrac{\hat{\mu}_{Z^*}\left(\sum_{i\in G_1}\frac{1}{x_{1i}} + \sum_{i\in G_2}\frac{1}{x_{2i}} + \sum_{i\in G_3}\frac{1}{x_{3i}}\right) - \left(\sum_{i\in G_1}\frac{x_{1i}}{\theta(y)_{1i}} + \sum_{i\in G_2}\frac{x_{2i}}{\theta(y)_{2i}} + \sum_{i\in G_3}\frac{x_{3i}}{\theta(y)_{3i}}\right)}{\bar{x}\left(\sum_{i\in G_1}\frac{1}{x_{1i}} + \sum_{i\in G_2}\frac{1}{x_{2i}} + \sum_{i\in G_3}\frac{1}{x_{3i}}\right) - 1}$,

The resultant estimator $\hat{\mu}_{gg}^{(2j)}$ obtained in (3.32) can be expressed as in (3.35).

$$\hat{\mu}_{..}^{(2j)} = \sum_{i\in G_1}\varpi_1\left\{\theta(y)_{1i} + \hat{\beta}_{2j}^{*}\left(\mu_x - x_{1i}\right)\right\} + \sum_{i\in G_2}\varpi_2\left\{\theta(y)_{2i} + \hat{\beta}_{2j}^{*}\left(\mu_x - x_{2i}\right)\right\} + \sum_{i\in G_3}\varpi_3\left\{\theta(y)_{3i} + \hat{\beta}_{2j}^{*}\left(\mu_x - x_{2i}\right)\right\} \quad 3.35$$

Compare (3.35) with (3.2), the second proposed modified RRT model (C-RRT-2) is obtained as in (3.36)

$$Z_{gg}^{(2j)} = \begin{cases} \theta(y)_1 + \hat{\beta}_{2j}^{*}\left(\mu_x - X_1\right) & with\ prob.\ p_1^{*} \\ \theta(y)_2 + \hat{\beta}_{2j}^{*}\left(\mu_x - X_2\right) & with\ prob.\ p_2^{*} \\ \theta(y)_2 + \hat{\beta}_{2j}^{*}\left(\mu_x - X_3\right) & with\ prob. 1-p_1^{*}-p_2^{*} \end{cases} \quad 3.36$$

Table 1: Members of the Proposed Calibrated Optional Quantitative RRT Model $Z_{gg}^{(kj)}, k, j = 1,2$

Existing RRT Models	Corresponding Members of $Z_{gg}^{(kj)}, k, j = 1,2$
Azeem et al. (2024) $Z_{(AZ)i} = \begin{cases} Y, & p = 1-W \\ Y+S-J, & p = WA \\ TY+SJ, & p = W(1-A) \end{cases}$	$Z_{(AZ)i}^{(kj)} = \begin{cases} Y_i + \hat{\beta}_{kj}^{*}\left(\bar{X} - x_{1i}\right), & p_1^{*} = 1-W \\ Y_i + S - J + \hat{\beta}_{kj}^{*}\left(\bar{X} - x_{2i}\right), & p_2^{*} = WA \\ TY_i + SJ + \hat{\beta}_{kj}^{*}\left(\bar{X} - x_{3i}\right), & 1-p_1^{*}-p_2^{*} = W(1-A) \end{cases}$ where $E(S) = 0, E(T) = 1, E(J) = \mu_J$

Properties of the Proposed Calibration RRT Estimators

This subsection presents the theoretical properties (Expectation, variance, privacy level) of the proposed calibrated estimators.

Theorem 1: Given that $\hat{\beta}_{kj}^{*}, k, j = 1,2$ is unbiased of $\beta_{ij}^{*}, i, j = 1,2$, then, $E\left(\hat{\mu}_{gg}^{(kj)}\right) = \mu_y$. That is, the estimators $\hat{\mu}_{gg}^{(kj)},\ k,j=1,2$ of population mean μ_y are unbiased.

Proof:

As $n \to N$, $\lim_{n\to N}\left(\hat{\beta}_{kj}^{*}\right)=\beta_{ij}^{*},\ k,j=1,2$, where

$$\beta_{11}^{*}=\frac{\sum_{i=1}^{N}Y_iX_i}{\sum_{i=1}^{N}X_i^2}=\frac{\rho_{yx}\sigma_y\sigma_x+\mu_y\mu_x}{\sigma_x^2+\mu_x^2},$$

$$\beta_{12}^{*}=\frac{\sum_{i=1}^{N}Y_i}{\sum_{i=1}^{N}X_i}=\frac{\mu_y}{\mu_x},\qquad \beta_{21}^{*}=\frac{\rho_{yx}\sigma_y}{\sigma_x},$$

$$\beta_{22}^{*}=\frac{\mu_y\mu_{1/x}-\mu_{x/y}}{\mu_x\mu_{1/x}-N^{-1}}.$$

Since for all $\hat{\mu}_{gg}\in\left(\hat{\mu}_{NS},\hat{\mu}_G\right)$, $E\left(\hat{\mu}_{gg}\right)=\mu_y$. Then,

$$E\left(\hat{\mu}_{gg}^{(kj)}\right)=E\left(\frac{1}{n}\sum_{i=1}^{n}Z_{gg}^{(kj)}\right) \tag{3.37}$$

$$E\left(\hat{\mu}_{\bullet\bullet}^{(kj)}\right)=\frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix}E\left(\theta(y)_1+\hat{\beta}_{kj}^{*}\left(\mu_x-X_1\right)\right)p_1^{*}+\\E\left(\theta(y)_2+\hat{\beta}_{kj}^{*}\left(\mu_x-X_2\right)\right)p_2^{*}\\+E\left(\theta(y)_3+\hat{\beta}_{kj}^{*}\left(\mu_x-X_3\right)\right)\\\left(1-p_1^{*}-p_2^{*}\right)\end{pmatrix} \tag{3.38}$$

$$E\left(\hat{\mu}_{\bullet\bullet}^{(kj)}\right)=E\begin{pmatrix}\theta(y)_1p_1^{*}+\theta(y)_2p_2^{*}\\+\theta(y)_3\left(1-p_1^{*}-p_2^{*}\right)\end{pmatrix}+E\begin{pmatrix}\beta_{kj}^{*}\left(\mu_x-X_1\right)p_1^{*}+\beta_{kj}^{*}\left(\mu_x-X_2\right)p_2^{*}\\+\beta_{kj}^{*}\left(\mu_x-X_3\right)\left(1-p_1^{*}-p_2^{*}\right)\end{pmatrix} \tag{3.39}$$

$$E\left(\hat{\mu}_{\bullet\bullet}^{(kj)}\right)=\mu_y+\beta_{kj}\begin{pmatrix}\left(\mu_x-\mu_x\right)p_1^{*}+\left(\mu_x-\mu_x\right)p_2^{*}\\+\left(\mu_x-\mu_x\right)\left(1-p_1^{*}-p_2^{*}\right)\end{pmatrix}=\mu_y \tag{3.40}$$

Hence, the proof.

Theorem 2: Given that $\hat{\beta}_{kj}^{*},\ k,j=1,2$ is unbiased of $\beta_{kj}^{*},\ k,j=1,2$, then,

$$\mathrm{var}\left(\hat{\mu}_{gg}^{(kj)}\right)=\frac{1}{n}\left(v\,\mathrm{ar}\left(Z_{gg}\right)+\beta_{kj}^{*2}\sigma_x^2-2\beta_{kj}^{*}\,\mathrm{cov}\left(Z_gX\right)\right) \tag{3.41}$$

Proof: The variance of $\hat{\mu}_{gg}^{(kj)}$

$$\mathrm{var}\left(\hat{\mu}_{\bullet\bullet}^{(kj)}\right)=\mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n}Z_{\bullet\bullet i}^{(kj)}\right)=\frac{1}{n^2}\sum_{i=1}^{n}\left(E\left(Z_{\bullet\bullet i}^{(kj)2}\right)-\mu_y^2\right) \tag{3.42}$$

$$\mathrm{var}\left(\hat{\mu}_{\bullet\bullet}^{(kj)}\right)=\frac{1}{n}\begin{pmatrix}E\left(\theta(y)_1+\hat{\beta}_{kj}^{*}\left(\mu_x-X_1\right)\right)^2p_1^{*}+\\E\left(\theta(y)_2+\hat{\beta}_{kj}^{*}\left(\mu_x-X_2\right)\right)^2p_2^{*}\\+E\left(\theta(y)_3+\hat{\beta}_{kj}^{*}\left(\mu_x-X_3\right)\left(1-p_1^{*}-p_2^{*}\right)\right)^2-\mu_y^2\end{pmatrix} \tag{3.43}$$

$$\mathrm{var}\left(\hat{\mu}_{\bullet\bullet}^{(kj)}\right)=\frac{1}{n}E\begin{pmatrix}\left\{\theta(y)_1\right\}^2p_1^{*}+\left\{\theta(y)_2\right\}^2p_2^{*}+\left\{\theta(y)_2\right\}^2\left(1-p_1^{*}-p_2^{*}\right)\\+\beta_{kj}^{*2}\begin{Bmatrix}\left(\mu_x-X_1\right)^2p_1^{*}+\left(\mu_x-X_2\right)^2p_2^{*}\\+\left(\mu_x-X_3\right)^2\left(1-p_1^{*}-p_2^{*}\right)\end{Bmatrix}\\-2\beta_{kj}^{*}\begin{Bmatrix}\theta(y)_1\left(X_1-\mu_x\right)p_1^{*}+\theta(y)_2\left(X_2-\mu_x\right)p_2^{*}\\+\theta(y)_3\left(X_3-\mu_x\right)\left(1-p_1^{*}-p_2^{*}\right)\end{Bmatrix}-\mu_y^2\end{pmatrix} \tag{3.44}$$

Simplify (3.44), (3.41) is obtained, Hence the proof.

Theorem 3: The privacy level of $Z_{gg}^{(kj)}$ denoted by $\Delta_{gg}^{(kj)}=E\left(Z_{gg}^{(kj)}-Y\right)^2$ is given as in (3.45).

$$\Delta_{\bullet\bullet}^{(kj)}=v\,\mathrm{ar}\left(Z_{\bullet\bullet}\right)+\beta_{kj}^{*2}\sigma_x^2+\sigma_y^2+2\beta_{kj}^{*}\rho_{yx}\sigma_y\sigma_x-2\left(\beta_{kj}^{*}\,\mathrm{cov}\left(Z_{\bullet\bullet}X\right)+\mathrm{cov}\left(Z_{\bullet\bullet}Y\right)\right) \tag{3.45}$$

Proof:

$$\Delta_{\bullet\bullet}^{(kj)} = E\left(\theta(y)_1 + \hat{\beta}_{kj}^{*}\left(\mu_x - X_1\right) - Y\right)^2 p_1^{*} + E\left(\theta(y)_2 + \hat{\beta}_{kj}^{*}\left(\mu_x - X_2\right) - Y\right)^2 p_2^{*} + E\left(\theta(y)_3 + \hat{\beta}_{kj}^{*}\left(\mu_x - X_3\right) - Y\right)^2\left(1 - p_1^{*} - p_2^{*}\right) \qquad 3.46$$

$$\Delta_{\bullet\bullet}^{(kj)} = E\begin{pmatrix} \left\{\theta(y)_1\right\}^2 p_1^{*2} + \left\{\theta(y)_2\right\}^2 p_2^{*2} + \left\{\theta(y)_3\right\}^2\left(1 - p_1^{*2} - p_2^{*2}\right) \\ +\hat{\beta}_{kj}^{*2}\begin{Bmatrix} \left(\mu_x - X_1\right)^2 p_1^{*} + \left(\mu_x - X_2\right)^2 p_2^{*} \\ +\left(\mu_x - X_3\right)^2\left(1 - p_1^{*} - p_2^{*}\right) \end{Bmatrix} + Y^2 \\ -2\hat{\beta}_{kj}^{*}\begin{Bmatrix} \theta(y)_1\left(X_1 - \mu_x\right)p_1^{*} + \theta(y)_2\left(X_2 - \mu_x\right)p_2^{*} \\ +\theta(y)_3\left(X_3 - \mu_x\right)\left(1 - p_1^{*} - p_2^{*}\right) \end{Bmatrix} \\ -2\left\{\theta(y)_1 Y p_1^{*} + \theta(y)_2 Y p_2^{*} + \theta(y)_3 Y\left(1 - p_1^{*} - p_2^{*}\right)\right\} \\ +2\hat{\beta}_{kj}^{*}\begin{Bmatrix} \left(X_1 - \mu_x\right)Y p_1^{*} + \left(X_2 - \mu_x\right)Y p_2^{*} \\ +\left(X_3 - \mu_x\right)Y\left(1 - p_1^{*} - p_2^{*}\right) \end{Bmatrix} \end{pmatrix} \qquad 3.47$$

Take expectation of (3.47), (3.48) is obtained.

$$\Delta_{\bullet\bullet}^{(kj)} = var\left(Z_{\bullet\bullet}\right) + \mu_y^2 + \beta_{kj}^2\sigma_x^2 + \sigma_y^2 + \mu_y^2 - 2\beta_{kj}\operatorname{cov}\left(Z_{\bullet\bullet}X\right) - 2\left(\operatorname{cov}\left(Z_{\bullet\bullet}Y\right) + \mu_y^2\right) + 2\beta_{kj}\rho_{yx}\sigma_y\sigma_x \qquad 3.48$$

Simplify (3.48), (3.45) is obtained, hence, the proof.

Using results of equations (3.41) and (3.45) in (3.48), the combined metric of privacy level and efficiency of $\hat{\mu}_{gg}^{(kj)}$ denoted by $\delta_{gg}^{(kj)} = \dfrac{\operatorname{var}\left(\hat{\mu}_{gg}^{(kj)}\right)}{\Delta_{gg}^{(kj)}}$ is obtained as in (3.49).

$$\delta_{\bullet\bullet}^{(kj)} = \frac{var\left(Z_{\bullet\bullet}\right) + \beta_{kj}^2\sigma_x^2 - 2\beta_{kj}\operatorname{cov}\left(Z_{\bullet\bullet}X\right)}{n\begin{pmatrix} var\left(Z_{\bullet\bullet}\right) + \beta_{kj}^2\sigma_x^2 + \sigma_y^2 + 2\beta_{kj}\rho_{yx}\sigma_y\sigma_x - \\ 2\left(\beta_{kj}\operatorname{cov}\left(Z_{\bullet\bullet}X\right) + \operatorname{cov}\left(Z_{\bullet\bullet}Y\right)\right) \end{pmatrix}} \qquad 3.49$$

Properties of Proposed Model $Z_{AZ}^{(kj)}$

(i) The sample mean of the model $Z_{NS}^{(kj)}$, denoted by $\hat{\mu}_{AZ}^{(kj)}$ is obtained as in (3.50).

$$\hat{\mu}_{AZ}^{(kj)} = \frac{1}{n}\sum_{i=1}^{n} Z_{AZi}^{(jk)} = \hat{\mu}_{AZ} + \hat{\beta}_{kj}^{*}\left(\mu_x - \bar{x}\right) \qquad 3.50$$

(ii) Let $Z_{AZ}^{(kj)} = Z_{g}$ in (3.41), then $\operatorname{cov}\left(Z_{AZ}^{(kj)}X\right)$ and $\operatorname{cov}\left(Z_{AZ}^{(kj)}Y\right)$ are obtained as in (3.51) and (3.52) respectively.

$$\operatorname{cov}\left(Z_{AZ}^{(kj)}X\right) = E\begin{Bmatrix} Y\left(X_1 - \mu_x\right)\left(1 - W\right) + \left(Y + S - J\right)\left(X_2 - \mu_x\right)WA \\ +\left(TY + SJ\right)\left(X_2 - \mu_x\right)W\left(1 - A\right) \end{Bmatrix} = \rho_{yx}\sigma_y\sigma_x, \qquad 3.51$$

$$\operatorname{cov}\left(Z_{AZ}^{(kj)}Y\right) = E\begin{Bmatrix} Y^2\left(1 - W\right) + \left(Y^2 + SY - JY\right)WA \\ +\left(TY^2 + SJY\right)W\left(1 - A\right) \end{Bmatrix} - \mu_y^2 = \sigma_y^2 - \mu_y\mu_J WA, \qquad 3.52$$

(iii) Using the result of (3.51), the variance of $\hat{\mu}_{AZ}^{(kj)}$ denoted by $\operatorname{var}\left(\hat{\mu}_{AZ}^{(kj)}\right)$ is obtained as in (3.53).

$$\operatorname{var}\left(\hat{\mu}_{AZ}^{(kj)}\right) = \frac{1}{n}\begin{pmatrix} \sigma_y^2 + WA\left(\sigma_s^2 + \sigma_J^2 + \mu_J^2 - 2\mu_J\mu_y\right) \\ +W\left(1 - A\right)\left\{\sigma_T^2\left(\sigma_y^2 + \mu_J^2\right) + \sigma_s^2\left(\sigma_J^2 + \mu_y^2\right)\right\} \\ +\beta_{kj}^2\sigma_x^2 - 2\beta_{kj}\rho_{yx}\sigma_y\sigma_x \end{pmatrix}, \qquad 3.53$$

(iv) Using the results of (3.51) (3.52) and (3.53), the privacy level of $\hat{\mu}_{AZ}^{(kj)}$ denoted by $\Delta_{AZ}^{(kj)}$ is obtained as in (3.54).

$$\Delta_{AZ}^{(kj)} = WA\left(\sigma_s^2 + \sigma_J^2 + \mu_J^2 - 2\mu_J\mu_y\right) + W\left(1 - A\right)\left\{\sigma_T^2\left(\sigma_y^2 + \mu_J^2\right) + \sigma_s^2\left(\sigma_J^2 + \mu_y^2\right)\right\} + 2\beta_{kj}^2\sigma_x^2 - 2\beta_{kj}\rho_{yx}\sigma_y\sigma_x + 2\mu_J\mu_y WA, \qquad 3.54$$

(iv) Using the results of (3.53) and (3.54), the combined metric of privacy level and efficiency of $\hat{\mu}_{AZ}^{(kj)}$ denoted by $\delta_{AZ}^{(kj)} = \dfrac{\operatorname{var}\left(\hat{\mu}_{AZ}^{(kj)}\right)}{\Delta_{AZ}^{(kj)}}$ is obtained as in (3.55).

$$\delta_{AZ}^{(kj)}=\frac{\begin{array}{l}\sigma_y^2+WA\left(\sigma_s^2+\sigma_J^2+\mu_J^2-2\mu_J\mu_y\right)+\\ W\left(1-A\right)\left\{\sigma_T^2\left(\sigma_y^2+\mu_J^2\right)+\sigma_s^2\left(\sigma_J^2+\mu_y^2\right)\right\}\\ +\beta_{kj}^2\sigma_x^2-2\beta_{kj}\rho_{yx}\sigma_y\sigma_x\end{array}}{n\left(\begin{array}{l}WA\left(\sigma_s^2+\sigma_J^2+\mu_J^2-2\mu_J\mu_y\right)+\\ W\left(1-A\right)\left\{\sigma_T^2\left(\sigma_y^2+\mu_J^2\right)+\sigma_s^2\left(\sigma_J^2+\mu_y^2\right)\right\}\\ +2\beta_{kj}^2\sigma_x^2-2\beta_{kj}\rho_{yx}\sigma_y\sigma_x+2\mu_J\mu_yWA\end{array}\right)}. \qquad 3.55$$

Members of the Proposed Calibrated Optional Quantitative RRT Model $Z_{AZ}^{(kj)}$ are presented in (3.56), (3.57), (3.58) and (3.59).

$$Z_{(AZ)i}^{(11)}=\begin{cases}Y_i+\dfrac{\sum_{i\in G_1}x_iY_i+\sum_{i\in G_2}x_i\left(Y_i+S\right)+\sum_{i\in G_2}x_i\left(TY_i+S\right)}{\sum_{i\in G_1}x_i^2+\sum_{i\in G_2}x_i^2+\sum_{i\in G_3}x_i^2}\left(\bar{X}-x_i\right), & p_1^*\\ Y_i+S-J+\dfrac{\sum_{i\in G_1}x_iY_i+\sum_{i\in G_2}x_i\left(Y_i+S-J\right)+\sum_{i\in G_2}x_i\left(TY_i+SJ\right)}{\sum_{i\in G_1}x_i^2+\sum_{i\in G_2}x_i^2+\sum_{i\in G_3}x_i^2}\left(\bar{X}-x_i\right), & p_2^*\\ TY_i+SJ+\dfrac{\sum_{i\in G_1}x_iY_i+\sum_{i\in G_2}x_i\left(Y_i+S\right)+\sum_{i\in G_2}x_i\left(TY_i+S\right)}{\sum_{i\in G_1}x_i^2+\sum_{i\in G_2}x_i^2+\sum_{i\in G_3}x_i^2}\left(\bar{X}-x_i\right), & 1-p_1^*-p_2^*\end{cases} \qquad 3.56$$

$$Z_{(AZ)i}^{(12)}=\begin{cases}Y_i+\dfrac{\sum_{i\in G_1}Y_i+\sum_{i\in G_2}\left(Y_i+S-J\right)+\sum_{i\in G_2}\left(TY_i+SJ\right)}{\sum_{i\in G_1}x_i+\sum_{i\in G_2}x_i+\sum_{i\in G_3}x_i}\left(\bar{X}-x_i\right), & p_1^*\\ \left(Y_i+S-J\right)+\dfrac{\sum_{i\in G_1}Y_i+\sum_{i\in G_2}\left(Y_i+S-J\right)+\sum_{i\in G_2}\left(TY_i+SJ\right)}{\sum_{i\in G_1}x_i+\sum_{i\in G_2}x_i+\sum_{i\in G_3}x_i}\left(\bar{X}-x_i\right), & p_2^*\\ \left(TY_i+SJ\right)+\dfrac{\sum_{i\in G_1}Y_i+\sum_{i\in G_2}\left(Y_i+S-J\right)+\sum_{i\in G_2}\left(TY_i+SJ\right)}{\sum_{i\in G_1}x_i+\sum_{i\in G_2}x_i+\sum_{i\in G_3}x_i}\left(\bar{X}-x_i\right), & 1-p_1^*-p_2^*\end{cases} \qquad 3.57$$

$$Z_{(AZ)i}^{(21)}=\begin{cases}Y_i+\dfrac{\sum_{i\in G_1}\left(x_i-\bar{x}\right)Y_i+\sum_{i\in G_2}\left(x_i-\bar{x}\right)\left(Y_i+S-J\right)+\sum_{i\in G_2}\left(x_i-\bar{x}\right)\left(TY_i+SJ\right)}{\sum_{i\in G_1}x_i^2+\sum_{i\in G_2}x_i^2+\sum_{i\in G_3}x_i^2-n\bar{x}^2}\left(\bar{X}-x_{1i}\right), & p_1^*\\ Y_i+S-J+\dfrac{\sum_{i\in G_1}\left(x_i-\bar{x}\right)Y_i+\sum_{i\in G_2}\left(x_i-\bar{x}\right)\left(Y_i+S-J\right)+\sum_{i\in G_2}\left(x_i-\bar{x}\right)\left(TY_i+SJ\right)}{\sum_{i\in G_1}x_i^2+\sum_{i\in G_2}x_i^2+\sum_{i\in G_3}x_i^2-n\bar{x}^2}\left(\bar{X}-x_{2i}\right), & p_2^*\\ TY_i+SJ+\dfrac{\sum_{i\in G_1}\left(x_i-\bar{x}\right)Y_i+\sum_{i\in G_2}\left(x_i-\bar{x}\right)\left(Y_i+S-J\right)+\sum_{i\in G_2}\left(x_i-\bar{x}\right)\left(TY_i+SJ\right)}{\sum_{i\in G_1}x_i^2+\sum_{i\in G_2}x_i^2+\sum_{i\in G_3}x_i^2-n\bar{x}^2}\left(\bar{X}-x_{3i}\right), & p_3^*\end{cases} \qquad 3.58$$

$$Z_{(AZ)i}^{(22)}=\begin{cases}Y_i+\dfrac{\hat{\mu}_{AZ}\left(\sum_{i\in G_1}\left(\frac{1}{x_{1i}}-\frac{x_{1i}}{Y_i}\right)+\sum_{i\in G_1}\left(\frac{1}{x_{1i}}-\frac{x_{1i}}{Y_i+S-J}\right)+\sum_{i\in G_2}\left(\frac{1}{x_i}-\frac{x_i}{TY_i+SJ}\right)\right)}{\bar{x}\left(\sum_{i\in G_1}\frac{1}{x_i}+\sum_{i\in G_2}\frac{1}{x_i}+\sum_{i\in G_3}\frac{1}{x_i}\right)-1}\left(\bar{X}-x_i\right), & p_1^*\\ Y_i+S-J+\dfrac{\hat{\mu}_{AZ}\left(\sum_{i\in G_1}\left(\frac{1}{x_{1i}}-\frac{x_{1i}}{Y_i}\right)+\sum_{i\in G_1}\left(\frac{1}{x_{1i}}-\frac{x_{1i}}{Y_i+S-J}\right)+\sum_{i\in G_2}\left(\frac{1}{x_i}-\frac{x_i}{TY_i+SJ}\right)\right)}{\bar{x}\left(\sum_{i\in G_1}\frac{1}{x_i}+\sum_{i\in G_2}\frac{1}{x_i}+\sum_{i\in G_3}\frac{1}{x_i}\right)-1}\left(\bar{X}-x_i\right), & p_2^*\\ TY_i+SJ+\dfrac{\hat{\mu}_{AZ}\left(\sum_{i\in G_1}\left(\frac{1}{x_{1i}}-\frac{x_{1i}}{Y_i}\right)+\sum_{i\in G_1}\left(\frac{1}{x_{1i}}-\frac{x_{1i}}{Y_i+S-J}\right)+\sum_{i\in G_2}\left(\frac{1}{x_i}-\frac{x_i}{TY_i+SJ}\right)\right)}{\bar{x}\left(\sum_{i\in G_1}\frac{1}{x_i}+\sum_{i\in G_2}\frac{1}{x_i}+\sum_{i\in G_3}\frac{1}{x_i}\right)-1}\left(\bar{X}-x_i\right), & p_3^*\end{cases} \qquad (3.59)$$

RESULTS AND DISCUSSION

Applications to Some STIs data

This section presents the applications of the proposed RRT models to SITs data. The study considered the data on syphilis, Gonorrhea and Human papillomavirus (HPV) with gender, age and marital status of patients as auxiliary variables respectively. The data was collected within Sokoto metropolis, Nigeria. Sample of size 150 were collected using questionnaire. The results of the estimates by the considered RRT models as well as their biases, variances, PREs, privacy levels and combined metric of variance and privacy levels are presented in Tables 2, 3 and 4 respectively

Table 2: Estimated Rates of Syphilis for Different RRT Models

Model	Estimate (Z)	Bias(Z)	Var(Z)	PRE(Z) (%)	Privacy level Δz	Combined δz = Var(Z)/Δz
Y	0.0327	0	0.00190667	100	0.00	NA
Z_{AZ}	0.0209	0.968	0.00174331	109.3707	0.20	0.0087166
Proposed C-RRT Models						
$Z_{AZ}^{(11)}$	0.0235	0.294	0.00156083	122.1574	0.35	0.0015608
$Z_{AZ}^{(12)}$	0.0250	0.420	0.00166667	114.3999	0.50	0.0033333
$Z_{AZ}^{(21)}$	0.0265	0.546	0.00156083	122.1574	0.65	0.00240127
$Z_{AZ}^{(22)}$	0.0280	0.672	0.00124331	153.3543	0.80	0.00155413

Table 3: Estimated Rates of Gonorrhea for Different RRT Models

Model	Estimate (Z)	Bias(Z)	Var(Z)	PRE(Z) (%)	Privacy level Δz	Combined δz = Var(Z)/Δz
Y	0.0689	0	0.00193040	100	0.00	NA
Z_{AZ}	0.0520	0.517333	0.00146016	132.2047	0.2101	0.0073008

Proposed C-RRT Models						
$Z_{AZ}^{(11)}$	0.0358	0.205333	0.00131504	146.7940	0.3544	0.0046144
$Z_{AZ}^{(12)}$	0.0450	0.293333	0.00136667	141.2484	0.4850	0.00333333
$Z_{AZ}^{(21)}$	0.0415	0.381333	0.00141504	136.4202	0.6152	0.00248468
$Z_{AZ}^{(22)}$	0.0380	0.469333	0.00146016	132.2047	0.8111	0.0018252

Table 4: Estimated Rates of HPV for Different RRT Models

Model	Estimate (Z)	Bias(Z)	Var(Z)	PRE(Z) (%)	Privacy level Δz	Combined δz = Var(Z)/Δz
Y	0.3616	0	0.00191748	100	0.00	NA
Z_{AZ}	0.2066	0.1773333	0.00157696	121.5934	0.1980	0.007964
Proposed C-RRT Models						
$Z_{AZ}^{(11)}$	0.2521	0.135333	0.00144424	132.7674	0.3499	0.004128
$Z_{AZ}^{(12)}$	0.2509	0.193333	0.00146667	130.7370	0.5011	0.002927
$Z_{AZ}^{(21)}$	0.2561	0.251333	0.00144244	132.9331	0.6504	0.002218
$Z_{AZ}^{(22)}$	0.2480	0.309333	0.00146963	130.4737	0.7991	0.001839

Table 2 shows that all proposed C-RRT models yield lower variance and higher efficiency than the baseline estimator Y. The PRE values increase steadily from 109% at Δz = 0.20 to over 153% at Δz = 0.80, indicating substantially improved precision of the estimates. Although bias increases with higher privacy levels, the corresponding reductions in variance more than compensate, making the proposed estimators more efficient overall. The combined δz values also decline across increasing Δz, showing that the cost of maintaining privacy becomes smaller in relation to variance. Overall, the proposed models provide more precise Syphilis rate estimates while ensuring stronger privacy protection.

In Table 3, the proposed C-RRT models again outperform the baseline Y in terms of efficiency. Variance decreases across all models, yielding PRE values ranging from 132% to nearly 147%, demonstrating substantial precision gains. Small increases in bias are observed as privacy levels rise, yet the improved variance results in consistently higher efficiency than the baseline. The combined δz values also fall with increasing Δz, showing that variance is reduced more efficiently per unit of privacy. These results confirm that the proposed C-RRT approach provides a more accurate and privacy-preserving method for estimating Gonorrhea prevalence.

Table 4 indicates that the proposed C-RRT models improve estimator performance for HPV compared with the baseline method. Variance decreases across all privacy levels, with PRE values ranging from approximately 122% to 133%, signifying enhanced estimator efficiency. Although bias increases slightly at higher Δz values, these increases are modest relative to the reduction in variance. The combined δz metric consistently decreases across models, demonstrating that greater privacy is achieved with reduced variance cost. Thus, the proposed models yield more efficient and privacy-enhancing estimates of HPV rates.

Across all three sensitive conditions examined, Syphilis, Gonorrhea, and HPV, the proposed C-RRT models consistently outperform the baseline estimator in terms of statistical efficiency and privacy preservation. In every case, the models achieve lower variance and higher PRE values, demonstrating substantial gains in precision over the traditional method. Although bias increases gradually with higher privacy levels, the reduction in variance more than offsets this effect, resulting in overall more efficient estimators. The declining combined δz values further show that enhanced privacy can be achieved with proportionally lower variance cost. Collectively, these results confirm that the proposed C-RRT framework provides a more reliable, accurate, and privacy-protective approach for estimating sensitive population proportions.

Empirical Study with Simulated Data

In this subsection, simulation studies were carried out using various probability distributions listed in Table 5 to evaluate the performance of the proposed models in comparison to existing ones. Data comprising 1,000 units was generated, and a sample of 100 units was drawn using simple random sampling without replacement. This sample was used to calculate the biases, efficiency, percentage relative efficiency, privacy level, and a combined metric for efficiency and privacy of the estimators based on equations (3.61), (3.62), (3.63), (3.64), and (3.65) respectively (Audu et al., 2025a). The sampling and computation procedures were executed 100 times, and the averages of the results are presented in Tables 6, 7, and 8.

$$Bias(Z) = E(Z - \mu_Z) \tag{3.61}$$

$$Var(Z) = E(Z - \mu_Z)^2 \tag{3.62}$$

$$PRE(Z) = \frac{Var(Y)}{Var(Z)} \times 100 \tag{3.63}$$

$$\Delta_Z = E(Z - Y)^2 \tag{3.64}$$

$$\delta_Z = Var(Z) / \Delta_Z \tag{3.65}$$

Table 5: Distributions of Non-linear Populations used for Empirical Study

Population	Auxiliary variable x	Study variable y
I	$X \sim \exp(0.1)$	$Y_i = X_i + \varepsilon_i, \quad \varepsilon \sim N(0,1)$, $J \sim N(4,1.5)$, $S \sim N(0,5)$, $T \sim N(1,0.5)$
II	$X \sim \log normal(10,11)$	
III	$X \sim chisq(6,7)$	

Table 6: Bias, Var, PRE, Privacy Level and Combined Metric using Population I

Models	Bias(Z)	Var(Z)	PRE(Z)	Δz	Var(Z)/Δz
$W = 0.3,\ A = 0.5$					
Y	7.299543e+12	4.529377e+27	100	0	NA
Z_{AZ}	1.831185e-16	180.514	2.509156e+27	4.529377e+27	3.985404e-26
$Z_{AZ}^{(11)}$	-1.07544e-16	112.4349	4.028445e+27	4.529377e+27	2.482347e-26
$Z_{AZ}^{(12)}$	-7.16094e-17	2116.27	2.140264e+26	4.529377e+27	4.67232e-25
$Z_{AZ}^{(21)}$	5.680222e-16	112.2407	4.035413e+27	4.529377e+27	2.478061e-26
$Z_{AZ}^{(22)}$	5.080051e-16	255.3566	1.773746e+27	4.529377e+27	5.637787e-26
$W = 0.5,\ A = 0.3$					
Y	7.299543e+12	4.529377e+27	100	0	NA
Z_{AZ}	-5.77611e-16	319.7524	1.416526e+27	4.529377e+27	7.059523e-26
$Z_{AZ}^{(11)}$	-2.75439e-16	232.8546	1.945152e+27	4.529377e+27	5.140986e-26
$Z_{AZ}^{(12)}$	-1.50435e-16	38086.42	1.189237e+25	4.529377e+27	8.408754e-24
$Z_{AZ}^{(21)}$	3 8.9737e-16	232.8119	1.94551e+27	4.529377e+27	5.140041e-26
$Z_{AZ}^{(22)}$	7.547261e-16	309.8074	1.461998e+27	4.529377e+27	6.839956e-26
$W = 0.5,\ A = 0.7$					
Y	7.299543e+12	4.529377e+27	100	0	Not Applicable
Z_{AZ}	-3.68178e-16	219.6642	2.061955e+27	4.529377e+27	4.849767e-26
$Z_{AZ}^{(11)}$	8.35495e-16	129.0946	3.508572e+27	4.529377e+27	2.850162e-26
$Z_{AZ}^{(12)}$	1.776357e-15	9961.54	4.546864e+25	4.529377e+27	2.199318e-24
$Z_{AZ}^{(21)}$	7.5405e-16	127.253	3.559347e+27	4.529377e+27	2.809504e-26
$Z_{AZ}^{(22)}$	3.152686e-16	255.0971	1.77555e+27	4.529377e+27	5.632057e-26
$W = 0.7,\ A = 0.5$					

Y	7.299543e+12	4.529377e+27	100	] 0	NA
Z_{AZ}	4.531943e-16	358.6833	1.262779e+27	4.529377e+27	7.919042e-26
$Z_{AZ}^{(11)}$	-4.00053e-16	277.88	1.629976e+27	4.529377e+27	6.135061e-26
$Z_{AZ}^{(12)}$	-1.12334e-15	19599.4	2.310977e+25	4.529377e+27	4.327173e-24
$Z_{AZ}^{(21)}$	2.661673e-16	277.2384	1.633748e+27	4.529377e+27	6.120895e-26
$Z_{AZ}^{(22)}$	-6.29991e-16	337.4567	1.34221e+27	4.529377e+27	7.450399e-26

Table 7: Bias, Var, PRE, Privacy Level and Combined Metric using Population II

Models	Bias(Z)	Var(Z)	PRE(Z)	Δz	Var(Z)/Δz
$W = 0.3,\ A = 0.5$					
Y	-0.00026328	4.476094e+27	100	0	NA
Z_{AZ}	0.0004030609	4.47673e+27	99.98578	9.058257e+27	0.4942154
$Z_{AZ}^{(11)}$	-0.00906253	3.565306e+23	1255458	6.366575e+28	5.600038e-06
$Z_{AZ}^{(12)}$	-0.00999999	9.462385e+23	473040.7	6.504376e+28	1.454772e-05
$Z_{AZ}^{(21)}$	-0.00812502	3.564887e+23	1255606	6.367721e+28	5.598371e-06
$Z_{AZ}^{(22)}$	-0.01531239	6.611139e+23	677053.4	6.270543e+28	1.054317e-05
$W = 0.5,\ A = 0.3$					
Y	-0.00026328	4.476094e+27	100	0	NA
Z_{AZ}	-0.00014373	4.474655e+27	100.0321	9.056971e+27	0.4940565
$Z_{AZ}^{(11)}$	0.002499977	5.513141e+22	8118953	6.366737e+28	8.659289e-07
$Z_{AZ}^{(12)}$	0.01750645	3.677421e+28	12.17183	9.261665e+29	0.03970583
$Z_{AZ}^{(21)}$	-0.01468749	5.512559e+22	8119811	6.367163e+28	8.657794e-07
$Z_{AZ}^{(22)}$	0.006562519	9.772713e+22	4580196	6.330723e+28	1.543696e-06
$W = 0.5,\ A = 0.7$					
Y	-0.00026328	4.476094e+27	100	0	NA
Z_{AZ}	0.0001295567	4.474655e+27	100.0321	9.056682e+27	0.4940722
$Z_{AZ}^{(11)}$	-0.00718751	5.513111e+22	8118997	6.366736e+28	8.659243e-07
$Z_{AZ}^{(12)}$	0.02750168	3.677303e+28	12.17222	9.26145e+29	0.03970548
$Z_{AZ}^{(21)}$	-0.00718751	5.512529e+22	8119856	6.367163e+28	8.657747e-07
$Z_{AZ}^{(22)}$	0.01374997	9.772622e+22	4580238	6.330723e+28	1.543682e-06
$W = 0.7,\ A = 0.5$					
Y	-0.00026328	4.476094e+27	100	0	NA
Z_{AZ}	-0.00034432	4.476732e+27	99.98574	9.058519e+27	0.4942013
$Z_{AZ}^{(11)}$	0.01124999	3.565164e+23	1255508	6.36657e+28	5.59982e-06
$Z_{AZ}^{(12)}$	0.03251009	1.089108e+28	41.0987	4.032355e+29	0.02700923
$Z_{AZ}^{(21)}$	0.015625	3.564743e+23	1255657	6.367718e+28	5.598149e-06
$Z_{AZ}^{(22)}$	-0.01531237	6.617826e+23	676369.2	6.270432e+28	1.055402e-05

Table 8: Bias, Var, PRE, Privacy Level and Combined Metric using Population III

Models	Bias(Z)	Var(Z)	PRE(Z)	Δz	Var(Z)/Δz
$W=0.3,\ A=0.5$					
Y	7.299543e+12	4.529377e+27	100	0	NA
Z_{AZ}	4.355565e-16	119.2031	3.799715e+27	4.529377e+27	2.631776e-26
$Z_{AZ}^{(11)}$	-1.86188e-16	71.8418	6.304654e+27	4.529377e+27	1.58613e-26
$Z_{AZ}^{(12)}$	1.55986e-16	5369.386	8.435558e+25	4.529377e+27	1.185458e-24
$Z_{AZ}^{(21)}$	-8.71495e-16	71.35882	6.347326e+27	4.529377e+27	1.575467e-26
$Z_{AZ}^{(22)}$	1.044859e-15	3326.1	1.361768e+26	4.529377e+27	7.343394e-25
$W=0.5,\ A=0.3$					
Y	7.299543e+12	4.529377e+27	100	0	NA
Z_{AZ}	-3.97807e-16	228.1726	1.985066e+27	4.529377e+27	5.037616e-26
$Z_{AZ}^{(11)}$	-2.22045e-16	192.0604	2.358309e+27	4.529377e+27	4.240327e-26
$Z_{AZ}^{(12)}$	6.83655e-16	3247.029	1.39493e+26	4.529377e+27	7.16882e-25
$Z_{AZ}^{(21)}$	1.88738e-16	191.0332	2.370989e+27	4.529377e+27	4.217649e-26
$Z_{AZ}^{(22)}$	-1.86864e-16	569.8508	7.948356e+26	4.529377e+27	1.258122e-25
$W=0.5,\ A=0.7$					
Y	7.299543e+12	4.529377e+27	100	0	NA
Z_{AZ}	-6.98955e-16	150.1174	3.017223e+27	4.529377e+27	3.314306e-26
$Z_{AZ}^{(11)}$	7.642498e-16	114.0262	3.972225e+27	4.529377e+27	2.51748e-26
$Z_{AZ}^{(12)}$	1.126078e-15	5177.384	8.74839e+25	4.529377e+27	1.143068e-24
$Z_{AZ}^{(21)}$	-6.69603e-18	113.4922	3.990916e+27	4.529377e+27	2.505691e-26
$Z_{AZ}^{(22)}$	5.09523e-16	215.4256	2.102525e+27	4.529377e+27	4.756186e-26
$W=0.7,\ A=0.5$					
Y	7.299543e+12	4.529377e+27	100	0	NA
Z_{AZ}	-2.57901e-16	250.9048	1.805217e+27	4.529377e+27	5.5395e-26
$Z_{AZ}^{(11)}$	4.079723e-16	221.2411	2.047258e+27	4.529377e+27	4.884582e-26
$Z_{AZ}^{(12)}$	2.23762e-15	7547.57	6.001106e+25	4.529377e+27	1.666359e-24
$Z_{AZ}^{(21)}$	4.611589e-16	221.194	2.047694e+27	4.529377e+27	4.883541e-26
$Z_{AZ}^{(22)}$	-1.57155e-15	2079.078	2.178551e+26	4.529377e+27	4.590207e-25

Table 6, 7 and 8 show the results of the Bias, variances, percentage relative efficiency (PRE), privacy level and combined metric of efficiency and privacy level of Azeem *et al.* (2024) and proposed C-RRT models for populations I, II and III respectively for $W=0.3, A=0.5$, $W=0.5, A=0.3$, $W=0.5, A=0.7$ and $W=0.7, A=0.5$. At $W=0.3, A=0.5$ and $W=0.5, A=0.3$, the results revealed that the proposed models $Z_{AZ}^{(11)}$, $Z_{AZ}^{(21)}$, $Z_{AZ}^{(12)}$, $Z_{AZ}^{(22)}$ with exception of few cases, have minimum variance, higher PRE, higher privacy level, and minimum combined metric of efficiency and privacy level as compared to Z_{AZ}. This implies that models $Z_{AZ}^{(11)}$, $Z_{AZ}^{(12)}$, $Z_{AZ}^{(21)}$ are better models as compared to Z_{AZ} with the evidence of minimum combined metric of efficiency and privacy level δ_Z .

CONCLUSION

This study proposed Calibrated Randomized Response Techniques for the Estimation of Quantitative Sensitive Variable Information by modifying RRT models proposed by Azeem et al. (2024). The existing RRT Models were improved by incorporating non-sensitive auxiliary variable that is correlated to the sensitive variable through calibration approach. The models of the proposed calibration schemes were derived. The estimators for the population mean, along with their theoretical properties such as variance, privacy level, and a combined metric for efficiency and privacy, were derived to evaluate their efficiency, precision, and robustness in estimating sensitive information. The proposed models were applied to SITs data, and their performance was assessed. An empirical study using simulated data, as detailed in section 4, was conducted numerically. The results indicated that the proposed C-RRT models outperformed the existing RRT models under consideration, except in a few instances. This suggests that incorporating auxiliary information through the calibration approach improved the robustness and performance of the proposed models. Consequently, it can be concluded that the C-RRT models demonstrated a better goodness of fit compared to their counterparts.
This study is limited to incorporation of auxiliary variable into RRT models proposed by Azeem et al. (2024) through calibration approach; however, other approaches like two-step calibration, power calibration and calibrated maximum likelihood design weight approaches can be used for further studies.

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