Journal of Basics and Applied Sciences Research (JOBASR)
ISSN (print): 3026-9091, ISSN (online): 1597-9962
Volume 1(1) IPSCFUDMA 2025 Special Issue
DOI: https://dx.doi.org/10.4314/jobasr.v1i1.14s

Novel Exact Soliton Solutions for the Extended (3+1)-dimensional Kairat –II Equation Using Two Robust Techniques

Ado Balili[1*] & Jamilu Sabi'u[2]
[1,2]Department of Mathematics, Northwest University, Kano.
*Corresponding Author Email: adobalili@gmail.com

ABSTRACT

This research explores the extended (3+1)-dimensional Kairat–II equation by employing the logistic equation method and the modified Kudryashov approach. The Kairat family of equations is notable for capturing second-order spatiotemporal dispersion and group velocity dispersion, which are significant in the study of curve differential geometry and various equivalence relations. The extended form of the equation introduces three additional linear diffusion terms, enriching the physical phenomena already described by the original model. This work presents a range of solitary wave solutions, including their propagation patterns, by applying hyperbolic and trigonometric function-based solutions. These include multiple breather, kink, and other essential wave structures relevant to optical fiber technology, signal processing, and telecommunication systems. Additionally, 3D, 2D, contour and polar coordinate plots are provided to visually represent the analytical soliton dynamics within the extended Kairat equation. The obtained solutions offer new perspectives in fields such as optical communication, fiber optics, oceanography, and quantum mechanics.

Keywords:
Soliton solutions; eKairat-II equation; Logistic equation method; Modified Kudryashov method

INTRODUCTION

Nonlinear integrable partial differential equations (NLIPDEs) are important in many branches of mathematics and research because of their intricate mathematical structures and many practical applications (Gupta et al 2023; Wazwaz, 2024) NLIPDEs are being studied within the framework of soliton theory due to their ability to represent complex nonlinear events, mathematical sophistication, and physical applicability. It has been found that soliton solutions provide a correct explanation for a wide range of physical phenomena, such as plasma waves, water waves, and optical pulses in fibres (Ullah et al 2023; Saifullah et al 2024;Wazwaz, 2023).

NLIPDEs are often the source of soliton solutions. Analytical solutions are necessary to fully analyse dynamics and apply them to real-world problems (Ali et al 2023; (Javed et al 2024). Over time, numerous effective methods have been created to construct analytical solutions for NLIPDEs. Several scholars have used techniques such as the planner dynamical scheme (Javed et al., 2023), the modified Tanh method (Wang et al., 2016), the bilinear approach (Gu et al., 2023), the generalized Kudryashov technique (Kumar et al., 2023), Tanh-coth method (Balili, 2024), sine-cosine method (Balili, 2024), new extended direct algebra method (Yusuf et al., 2025), logistic equation and exponential rational function methods (Balili et al., 2024), Tanh-coth method (Bello et al., 2024), sine-cosine method (Muhammad et al., 2024) and sine-cosine method (Muhammad et al., 2025), the exponential rational function approach (Wazwaz et al 2023; Zhu et al 2024), separation of variable method (Zhu et al., 2023), generalized extended function method (Kai et al., 2022) and the logarithmic transformation approach (Zhu et al., 2023). The extensive literature on the issue of many types of soliton solutions, such as kink, dark, and periodic solitons (Parasuraman, 2023). Various authors have studied soliton solutions and chaotic behaviour in a nonlinear Schrödinger model with a random potential (Ali et al., 2024), regularized long-wave equation (Kai et al., 2022) and complex Ginzburg-Landau equation (Zhu et al., 2024).

In this study, we explore the complex evolutionary dynamics embedded within the Kairat-II model (Awadalla et al., 2023). This model characterizes the surface geometry of curves and has applications in various fields, including quantum mechanics, optical fiber systems, and optical communication technologies (Iqbal et al., 2024; Iqbal et al., 2024 and (Wazwaz, 2024).

The classical integrable form of the Kairat-II model is given as (Myrzakulova et al., 2023):

$$\mathcal{V}_{xt} + \mathcal{V}_{xxxt} - 2\mathcal{V}_t\mathcal{V}_{xx} - 4\mathcal{V}_x\mathcal{V}_{xt} = 0, \quad (1)$$

The main objective of this work is to investigate certain dynamic behaviors of the extended (3+1)-dimensional Kairat-II equation (eKairat- IIE) that have not been adequately addressed in the existing literature. These include utilizing logistic equation and modified Kudryashov methods (LEM and MKM) to construct the exact travelling wave solutions of (eKairat- IIE).

The extended (3+1)-dimensional Kairat-II equation is formulated as:

$$\mathcal{V}_{xxxt} - 4\mathcal{V}_x\mathcal{V}_{xt} + \mathcal{V}_{xt} - 2\mathcal{V}_t\mathcal{V}_{xx} + \beta_1\mathcal{V}_{xx} + \beta_2\mathcal{V}_{xy} + \beta_3\mathcal{V}_{xz} = 0. \quad (2)$$

Where $\mathcal{V}_x\mathcal{V}_{xt}$ and $\mathcal{V}_t\mathcal{V}_{xx}$ represent nonlinear terms, and β_1, β_2 and β_3 are positive real parameters.

The structure of the paper is organized as follows: Section **2** presents a concise overview of the LEM and MKM methods. Section **3** introduces the mathematical analysis for the solution of (eKairat-IIE). Section **4** Applications of the proposed methods are discussed. Section **5** presents results and graphical discussion. Section **6** concludes the paper by summarizing the key findings.

MATERIALS AND METHODS

PROPOSED SCHEMES

Analysis of the Logistic Equation Method (LEM)

In this section logistic equation method will be explained in details (Balili et al., 2024).

Let us consider the NLPDE

$$P\left(u, u_t, u_x, u_y, u_{xx}, \dots\right) = 0, \quad (3)$$

Where the function $u = u(x, y, z, t)$ represents an unknown function in the given context. In order to proceed, we have to introduce a wave transformation as follows:

$$u(x, y, z, t) = U(\xi), \text{ where } \xi = x + y + z - \varpi t \varpi \neq 0 \text{ (wave speed)} \quad (4)$$

Step1.Transform NPDE to ODE using Eq. (4) as

$$Q(U, U', U'', \dots) = 0, \quad (5)$$

Step 2. Assume that Eq. (5) has the following solution

$$U(\xi) = c_0 + \sum_{i=1}^{n} c_i N(\xi)^i, \quad c_n \neq 0, \quad (6)$$

Where c_i for $i = 0,1,2, \dots, n$ are constants, and $N(\xi)$ satisfies the following logistic equation:

$$N'(\xi) = s_0 N(\xi)\left(1 - \frac{N(\xi)}{s_1}\right), \quad (7)$$

Step 3. Impose the homogeneous balancing method on the highest order derivative of U and the highest order nonlinear term in the Eq. (5) and we can gain the integer n of Eq. (6). Applying Eq. (6) and Eq. (7) to Eq. (5), we are able to acquire an algebraic system through the coefficients of N with same order. Utilizing the results of the algebraic system, we can obtain the values of c_i, s_0 *and* s_1 and related restriction conditions.

Step 4. By substituting the results of the previous steps into Eq. (6) and applying the general solutions of Eq. (7) as follows:

$$N(\xi) = \frac{s_1}{1 + a s_1 \exp(-s_0\xi)}, \quad (8)$$

Provided $a, s_0,$ *and* s_1 are arbitrary constants, we can achieve closed form solutions of Eq. (5)

Analysis of the Modified Kudryashov Method (MKM)

The modified Kudryashov method involves the following steps in solving the nonlinear partial differential equation (NLPDE) (Hosseini, et al., 2017; Seadawy et al., 2018):

Step 1. Consider the given NLPDE of the following form $u = u(x, y, z, t)$:

$$\mathcal{P}\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, u_{yy}, u_{zz} \dots\right) = 0. \quad (9)$$

Step 2. Applying the wave transformation $u(x, y, z, t) = U(z)$, in Eq. (9), where :

$$z = \mu(x + y + z - \lambda t). \quad (10)$$

Here, μ is the wave variable and λ is the velocity; both are non-zero constants. Eq. (9) transforms to the following ordinary differential equation (ODE):

$$Q(U, U', U'', UU', \dots) = 0, \quad (11)$$

Where the prime denotes the derivative with respect to z.

Step 3. Let the initial solution of Eq. (11) assume to be,

$$U(z) = \sum_{i=0}^{N} a_i Q^i(z), \quad (12)$$

Where N is a non-zero and positive constant, calculated by the principle of homogeneous balancing of Eq. (11), $a_i; i = 0,1,2, \dots$ are unknowns to be determined and $Q(z)$ is the solution of the following auxiliary ODE:

$$\frac{dQ(z)}{dz} = Q(z)(Q(z) - 1).\ln(a); a \neq 1, \quad (13)$$

$$Q(z) = \frac{1}{1 \pm D a^z}, \quad (14)$$

Where D is the integral constant and we assume $D = 1$.

Step 4. Substituting Eqs. (12) - (13) in Eq. (11) leads to the polynomial in $(z)^i$; $i = 0,1,2, \dots$. As $Q(z)^i \neq 0$, so collecting its coefficients and then equating to zero gives the systems of overdetermined algebraic equations, which upon solving give the unknowns of Eq. (12) and Eq. (14).

Step 5. Finally, substituting the values of step 4 in Eq. (14) and then in Eq. (12) gives the solution $U(z)$ of Eq. (11).

MATHEMATICAL ANALYSIS (eKairat-IIE)

To extract the solution of Eq. (2), we start as follows:

Using the following wave transformation

$$\mathcal{V}(x, y, z, t) = \mathfrak{V}(\varrho), \varrho = x + y + z - \varpi t. \quad (15)$$

Where ϖ is the wave' speed and $\mathfrak{V}(\varrho)$ is a real function.

The Eq. (2) reduces to ordinary differential equation (ODE):

$$-\varpi\mathfrak{V}''' + 3\varpi(\mathfrak{V}')^2 + (\beta_1 + \beta_2 + \beta_3 - \varpi)\mathfrak{V}' = 0. \quad (16)$$

The prime above means the derivative w.r.t. ϱ .Eq. (16) is the reduced ODE.

RESULTS AND DISCUSSION

Exact Solutions of the extended (3+1) Dimensional Kairat II Equation (eKairat- IIE) Using Logistic Equation Method (LEM)

In this section, the solution of extended Kairat II (eKairat-IIE) using logistic equation method Eq. (2) will be discussed.
Now to solve Eq. (16) by applying (LEM).
By homogeneous balancing principle between $\mathfrak{V}''$ *and* $(\mathfrak{V}')^2$ one obtains the value of $n = 1$ in Eq. (16).
Assuming Eq. (16) has the following form:

$$\mathfrak{V}(\varrho) = c_0 + c_1 N(\varrho), \quad (17)$$

Where $N(\varrho)$ satisfy Eq. (7) with value Eq. (8).
By substituting Eq. (17) and its derivatives in Eq. (16), we obtain the following system of algebraic equations as follows:

$$N^4(\varrho): \quad 3c_1^2\varpi s_0^2 s_1 + 6c_1\varpi s_0^3 = 0, \quad (18)$$

$$N^3(\varrho): \quad -6c_1^2\varpi s_0^2 s_1^2 - 12c_1\varpi s_0^3 s_1 = 0, \quad (19)$$

$$N^2(\varrho): \quad 3c_1^2\varpi s_0^2 s_1^3 + 7c_1\varpi s_0^3 s_1^2 - \beta_1 c_1 s_0 s_1^2 - \beta_2 c_1 s_0 s_1^2 - \beta_3 c_1 s_0 s_1^2 + c_1\varpi s_0 s_1^2 = 0, \quad (20)$$

$$N(\varrho): \quad -c_1\varpi s_0^3 s_1^3 + \beta_1 c_1 s_0 s_1^3 + \beta_2 c_1 s_0 s_1^3 + \beta_3 c_1 s_0 s_1^3 - c_1\varpi s_0 s_1^3 = 0. \quad (21)$$

Solving the above system of algebraic equations, we get

$$c_0 = c_{0,} c_1 = -\frac{2s_0}{s_1},\ \varpi = \frac{\beta_1+\beta_2+\beta_3}{s_0^2+1}.$$

(22)
.
By inserting these values Eq. (22) into Eq. (17) and using the definition of $N(\varrho)$ (Eq. (8), the following solutions are extracted for Eq. (16).

$$U(\varrho) = \frac{a c_0 s_1 e^{s_0\varrho} + c_0 - 2s_0}{a s_1 e^{-s_0\varrho} + 1}. \quad (23)$$

Where $\varrho = x + y + z - \varpi t$.

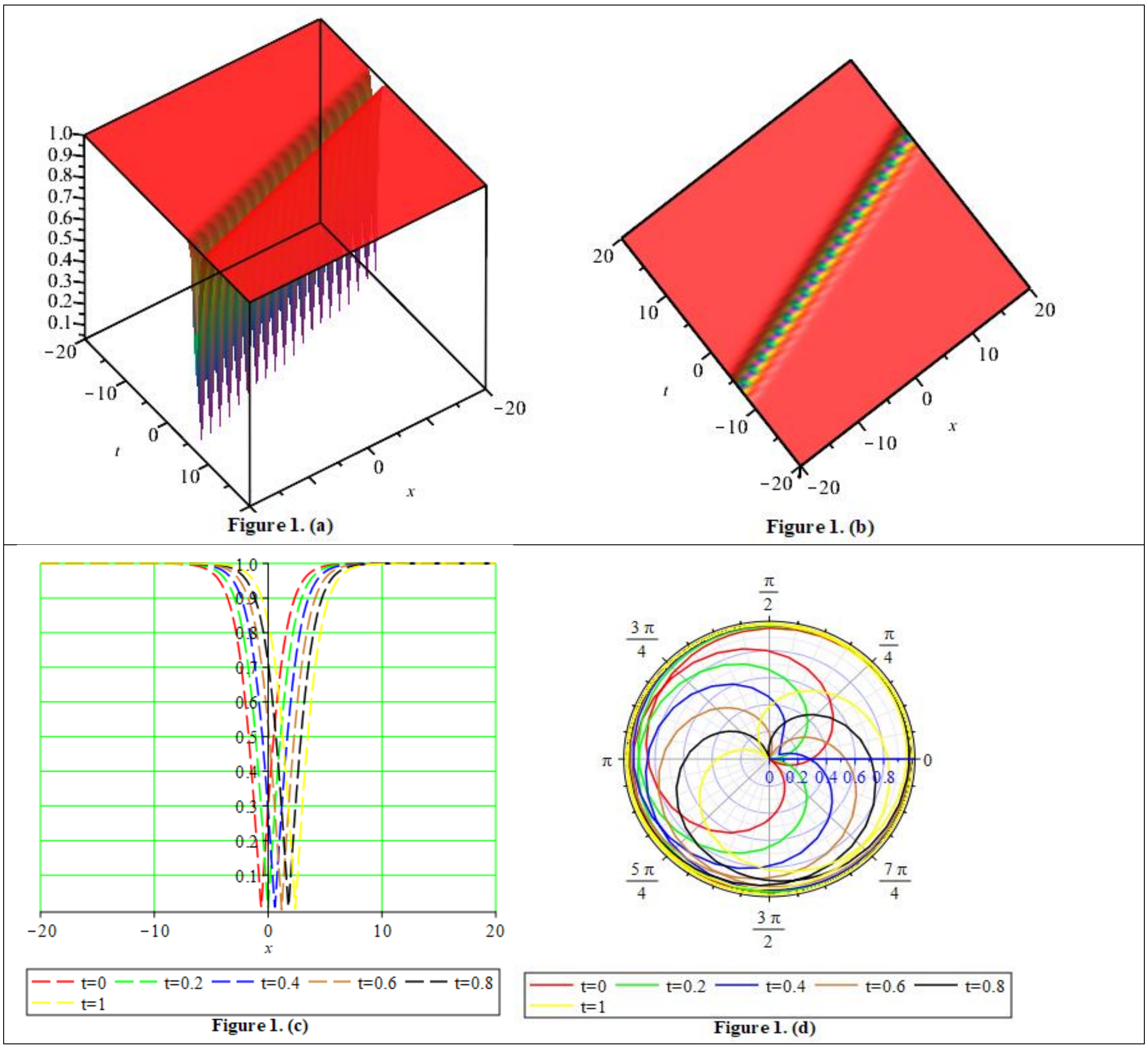

Figure 1. The graphical representation of $u_{1,1}(x, y, z, t)$, Eq.(23) with values $a = 2, c_0 = 1, c_1 = 2, s_0 = 1, s_1 = 2, \varpi = 3, \beta_1 = 2, \beta_2 = 2, \beta_3 = 2, y = 1, z = 1$.**(a)** $3D\ surface$, (***b***)$Contour\ plot$, (***c***)$2D\ slice$, (***d***)Polar coordinate plot

Exact Solutions for the extended (3+1) Dimensional Kairat II Equation (eKairat- IIE) Using Modified Kudryashov Method (MKM)

In this section, the solution of (eKairat- IIE**)** using (MKM) method will be explained.

Solving Eq. (16) by applying (MKM)

We assume the solution to be in this form:

$$\mathfrak{V}(\varrho) = n_0 + n_1\mathcal{M}(\varrho), \quad (24)$$

Where $\mathcal{M}(\varrho)$ satisfy Eq. (13) with value Eq. (14).

By substituting Eq. (24) into Eq. (16), we obtain the following system of equations as follows:

$$\mathcal{M}(\varrho)^4: -6\varpi n_1 \ln(a)^3 + 3\varpi n_1^2 \ln(a)^2 = 0, \quad (25)$$

$$\mathcal{M}(\varrho)^3: 12\varpi n_1 \ln(a)^3 - 6\varpi n_1^2 \ln(a)^2 = 0, \quad (26)$$

$$\mathcal{M}(\varrho)^2: -7\varpi n_1 \ln(a)^3 + 3\ln(a)^2 \varpi n_1^2 + \ln(a)\beta_1 n_1 + \ln(a)\beta_2 n_1 + \ln(a)\beta_3 n_1 - \ln(a)\varpi n_1 = 0,$$

(27)

$$\mathcal{M}(\varrho): \varpi n_1 \ln(a)^3 + \ln(a)\beta_1 n_1 - \ln(a)\beta_2 n_1 - \ln(a)\beta_3 n_1 + \ln(a)\varpi n_1 = 0.$$

(28)

Solving the above system of equations, we obtain

$$n_0 = n_0, n_1 = 2\ln(a), \varpi = \frac{\beta_1+\beta_2+\beta_3}{1+\ln(a)^2} \quad (29)$$

Inputting the above values in Eq. (34) we get the solutions

$$u_{2,1}(x, y, z, t) = n_0 + \frac{2\ln(a)}{1+Ca^{\varrho}}, \quad (30)$$

$$u_{2,2}(x, y, z, t) = n_0 + \frac{2\ln(a)}{1-Ca^{\varrho}}, \quad (31)$$

Where $\varrho = x + y + z - \varpi t.$

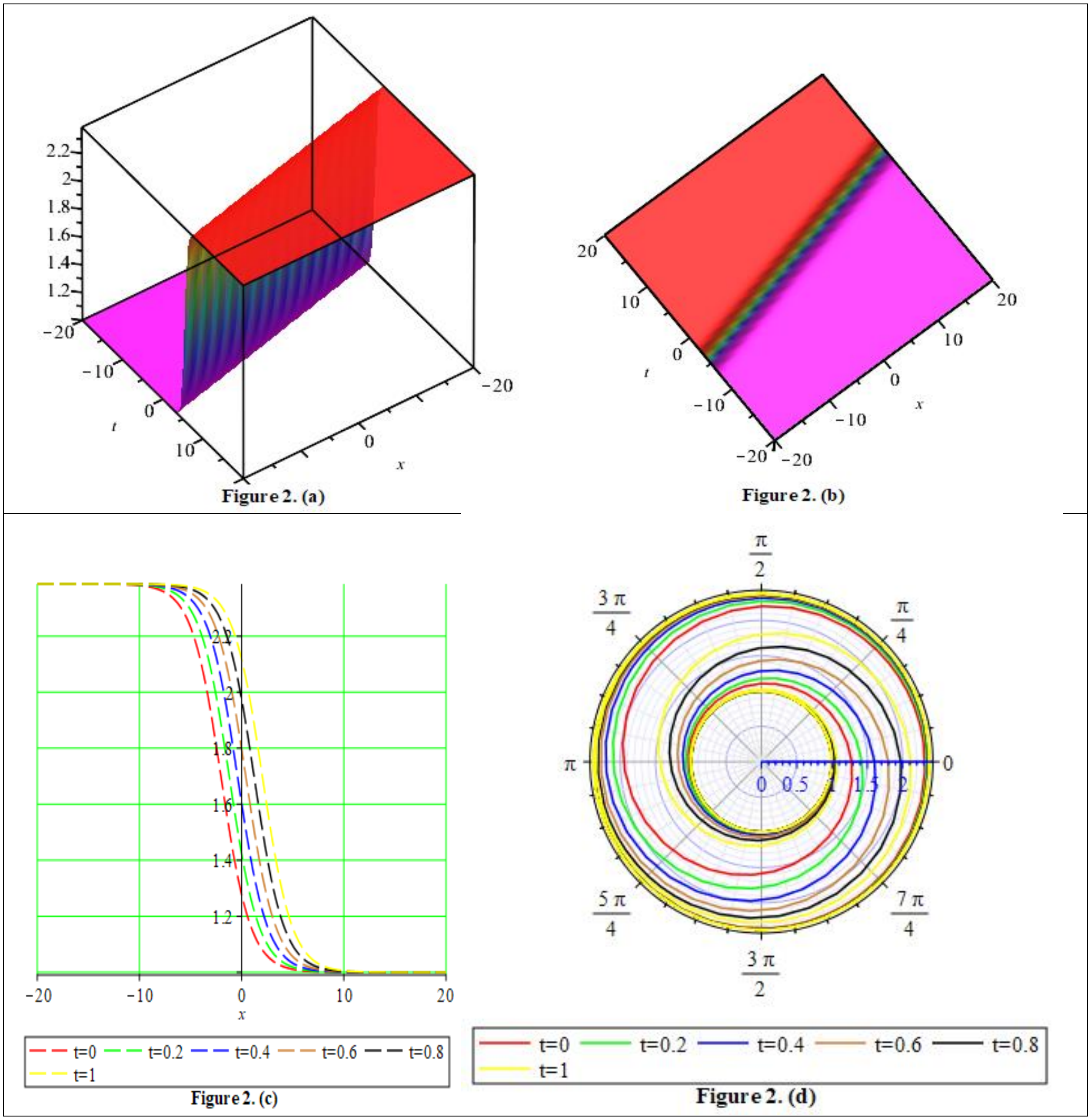

Figure 2. The graphical illustration of solution $u_{2,1}\ (x,y,z,t)$, Eq.(30) with values $a = 2, \mathcal{n}_0 = 1, C = 1, \varpi = \frac{6}{1+\ln(2)^2}, \beta_1 = 2, \beta_2 = 2, \beta_3 = 2, y = 1, z = 1$. **(a)** $3D\ plot$, $(\boldsymbol{b})Contour\ plot$, $(\boldsymbol{c})\ 2D$, $(\boldsymbol{d})$Polar coordinate plot.

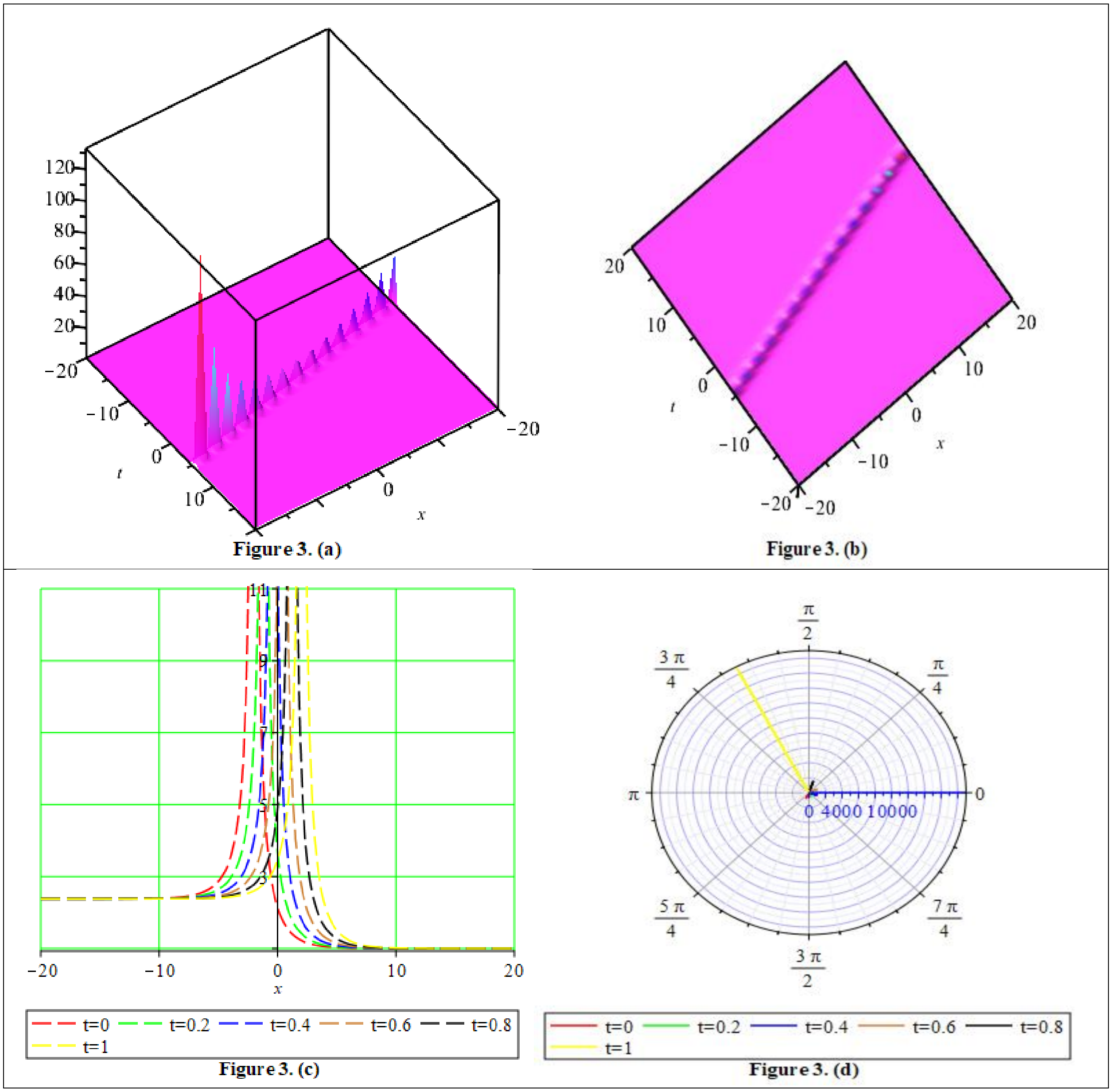

Figure 3. The graphical illustration of solution $u_{2,2}\,(x,y,z,t)$, Eq.(31) with values $a = 2, n_0 = 1, C = 1, \varpi = \frac{6}{1+\ln(2)^2}, \beta_1 = 2, \beta_2 = 2, \beta_3 = 2, y = 1, z = 1$. **(a)** $3D\ plot, (\boldsymbol{b})Contour\ plot, (\boldsymbol{c})2D, (\boldsymbol{d})$Polar coordinate plot.

The logistic equation method (LEM) and the modified Kudryashov Method (MKM) have been applied to construct exact travelling wave solutions for the extended Kairat IIE (eKairat -IIE) utilizing two analytical techniques: the Modified Kudryashov method (MKM) and the logistic equation method (LEM) with the aid of Maple 19 software. Several non-trivial solutions were acquired through both methods, presented in Eq. (23) for LEM and Eqs. (30) – (31) for MKM. Numerical simulations were demonstrated for the solutions from both methods.

Figures 1 through 3 illustrate various graphical representations of these solutions. In each figure: (a) shows the 3D surface plot, (b) displays the contour plot, (c) presents the 2D profile, and (d) provides the corresponding polar coordinate plot, highlighting the solution behavior under different parameter settings.

Figure 1. Presents the structures of solution $u_{1,1}(x,y,z,t)$, Eq. (23) which exhibits dark multiple solitons for $a = 2, c_0 = 1, c_1 = 2, s_0 = 1, s_1 = 2, \varpi = 3, \beta_1 = 2, \beta_2 = 2, \beta_3 = 2, y = 1, z = 1$.

Figure 2. Shows the solution $u_{2,1}\,(x,y,z,t)$, from Eq. (30), also illustrating kink solitons behaviour with parametr values $a = 2, n_0 = 1, C = 1, \varpi =$

$\frac{6}{1+\ln(2)^2}, \beta_1 = 2, \beta_2 = 2, \beta_3 = 2, y = 1, z = 1.$.

Figure 3. Depicts solution $u_{2,2}\ (x, y, z, t)$,with parameter values$a = 2, n_0 = 1, C = 1, \varpi = \frac{6}{1+\ln(2)^2}, \beta_1 = 2, \beta_2 = 2, \beta_3 = 2, y = 1, z = 1$. From Eq. (31), revealing multiple solitons structures with identical parameter values.

The extracted solutions encompass rational and exponential function forms. Their physical properties and dynamics are effectively visualized through the above mentioned graphs, showing the influence of varying parameter values on the solution profiles as depicted in Figures 1–3.

CONCLUSION

In this study, new solutions of the nonlinear extended Kairat II equation is obtained using the logistic equation method (LEM) and the modified Kudryashov Method (MKM). By assigning arbitrary values to the free parameters, a variety of solution types were obtained, including rational and exponential functions. The results testify that both techniques are effective and reliable. This work demonstrates the power and efficiency of the logistic equation method and the modified Kudryashov method in generating exact analytical solutions for a broad type of nonlinear equations. Furthermore, the approaches are adaptable and can be extended to solve other complex nonlinear models encountered in mathematical physics, fiber optics, and plasma physics to mention a few.

REFERENCES

Ali, A., Ahmad, J., Javed, S., Hussain, R., Alaoui, M.K. (2024). Numerical simulation and investigation of soliton solutions and chaotic behavior to a stochastic nonlinear Schrödinger model with a random potential. *Plos one 19*(1), e0296678.

Ali, Asghar, Ahmad, Jamshad, Javed, Sara (2023). Exploring the dynamic nature of soliton solutions to the fractional coupled nonlinear Schrödinger model with their sensitivity analysis. Opt. Quant. Electron. 55(9), 810.

Awadalla, M., Zafar, A., Taishiyeva, A., Raheel,M., Myrzakulov, R., Bekir, A. (2023). The analytical solutions to the M-fractional Kairat-II and Kairat-X equations. *Front. Phys. 11*, 1335423.

Balili, A. (2023). New Exact Solutions for Time-Beta Fractional Boussinesq-Like Equations, Using the Sine-Cosine Method. *Yumsuk, Journal of Pure and Applied Sciences, Conference Edition, 1* (1), 285-296.

Balili, A. (2023). Travelling Wave Solutions for the Space and Time β-Fractional Chen-Lee-Liu Equation Using Tanh- Coth Method. *Yumsuk, Journal of Pure and Applied Sciences, Conference Edition, 1* (1), 297-305.

Balili, A. and Sabi'u, J. (2024). New Exact Solutions for Space-Time β- Fractional Generalized PHI-Four Equation Using Exponential Rational Function and Logistic Equation Schemes, *Bayero Journal of Pure and Applied Sciences, Special Conference Edition, 15* (1), 248-253. http://dx.doi.org/10.4314/bajopas.v15i1.2S

Bello, H. N., Maitama, S., and Balili, A. (2024). Analytical soliton solutions for space-time fractional order nonlinear evolution equations with M-Truncated derivatives. *Bayero Journal of Pure and Applied Sciences, Special Conference Edition*, 15(1), 342-351. http://dx.doi.org/10.4314/bajopas.v15i1.53S

Gu, Y., Zia, S.M., Isam, M., Manafian, J., Hajar, A., Abotaleb, M. (2023). Bilinear method and semi-inverse variational principle approach to the generalized (2+ 1)-dimensional shallow water wave equation. *Res. Phys. 45*, 106213.

Gupta, R.K., Sharma, M. (2024). An extension to direct method of Clarkson and Kruskal and Painlevé analysis for the system of variable coefficient nonlinear partial differential equations. *Qual. Theory Dyn. Syst. 23*(3), 115.

Iqbal, M., Dianchen, L., Alammari, M., Seadawy, Aly R., Alsubaie, N. E., Umurzakhova, Z., Myrzakulov, R. (2024). A construction of novel soliton solutions to the nonlinear fractional Kairat-II equation through computational simulation. *Opt. Quant. Electron. 56*(5), 845.

Iqbal, M., Lu, D., Seadawy, A.R., Alomari, F.A., Umurzakhova, Z., Myrzakulov, R. (2024). Constructing the soliton wave structure to the nonlinear fractional Kairat-X dynamical equation under computational approach. *Mod. Phys. Lett. B 18*, 2450396.

Javed, S., Ali, A., Ahmad, J., Hussain, R. (2023). Study the dynamic behavior of bifurcation, chaos, time series analysis and soliton solutions to a Hirota model. *Opt. Quant. Electron. 55*(12), 1114.

Javed, Sara, Ali, Asghar, Muhammad, Taseer (2024). Dynamical perspective of bifurcation analysis and soliton solutions to (1+ 1)-dimensional nonlinear perturbed Schrödinger model. *Opt. Quant. Electron. 56*(6), 1013.

Kai, Y., Jialiang, J., Yin, Z. (2022). Study of the generalization of regularized long-wave equation. *Nonlinear Dyn. 107*(3), 2745–2752.

Kai, Y., Yin, Z. (2022). Linear structure and soliton molecules of Sharma-Tasso-Olver-Burgers equation. *Phys. Lett. A 452*, 128430.

Kumar, S., Niwas, M. (2023). Optical soliton solutions and dynamical behaviours of Kudryashov's equation employing efficient integrating approach. *Pramana 97*(3), 98.

Muhammad, M. S., Balili, A. and Sulaiman, S. (2024). New Exact Solutions of the Fractional Korteweg de Vries Benjami-Bona Mahoni and Modified Korteweg de Vries -Zakharov-Kuznetsov Equations Via a powerful sine-cosine method. *BAJOPAS* 15(1), 301-306. http://dx.doi.org/10.4314/bajopas.v15i1.46S

Muhammad, M. S., Balili, A. and Sulaiman, S. (2025). New Analytical Solution for the Fractional Modified Korteweg de Vries-Zakharov-Kuznetsov Equation with Beta Derivative. *UMYU Scientifica*, 4(2), 62-66. https://doi.org/10.56919/usci.2542.008

Myrzakulova, Zh., Manukure, S., Myrzakulov, R., Nugmanova, G. (2023). Integrability, geometry and wave solutions of some Kairat equations. arXiv preprint arXiv:2307.00027.

Parasuraman, E. (2023). Stability of kink, anti kink and dark soliton solution of nonlocal Kundu Eckhaus equation. *Optik 290*, 171279.

Saifullah, S., Ahmad, S., Khan, M.A., ur Rahman, M. (2024). Multiple solitons with fission and multi waves interaction solutions of a (3+ 1)-dimensional combined pKP-BKP integrable equation. *Phys. Scripta 99*(6), 065242.

Ullah, N., Asjad, M.I., Hussanan, A., Akgül, A., Alharbi,Wedad R., Algarni, H., Yahia, I.S. (2023) Novel wave structures for two nonlinear partial differential equations arising in the nonlinear optics via Sardar-sub equation method. *Alex. Eng. J. 71*, 105–113.

Wang, Y.Y., Zhang, Y.P., Dai, C.Q. (2016). Restudy on localized structures based on variable separation solutions from the modified tanh-function method. *Nonlinear Dyn. 83*(3), 1331–1339.

Wazwaz, A.M., Alhejaili, W., El-Tantawy, S.A. (2023). Analytical study on two new (3+ 1)-dimensional Painlevé integrable equations: kink, lump, and multiple soliton solutions in fluid mediums. *Phys. Fluids 35*(9), 093119.

Wazwaz, Abdul-Majid. (2024). A Hamiltonian equation produces a variety of Painlevé integrable equations: solutions of distinct physical structures. Int. J. Num. Methods Heat Fluid Flow 34(4), 1730–1751.

Wazwaz, Abdul-Majid. (2024). Extended (3+ 1)-dimensional Kairat-II and Kairat-X equations: Painlevé integrability, multiple soliton solutions, lump solutions, and breather wave solutions. *Int. J. Num. Methods Heat Fluid Flow 34*(5), 2177–2194.

Wazwaz, Abdul-Majid., Gui-Qiong, X. (2023). Variety of optical solitons for perturbed Fokas-Lenells equation through modified exponential rational function method and other distinct schemes. *Optik 287*, 171011.

Yusuf, G., Sabi'u, J., Ibrahim, S. I., Balili, A. (2025). Wave propagation patterns for the (2+1)-dimensional modified complex KdV system via new extended direct algebra method. *Journal of basics and applied sciences research, 3*(2), 53-61.

Zhu, C., Al-Dossari, M., Rezapour, S., Alsallami, S.A.M., Gunay, B. (2024). Bifurcations, chaotic behavior, and optical solutions for the complex Ginzburg-Landau equation. *Res. Phys. 59*, 107601.

Zhu, C., Al-Dossari, M., Rezapour, S., Gunay, B. (2024). On the exact soliton solutions and different wave structures to the (2+ 1) dimensional Chaffee-Infante equation. *Res. Phys. 57*, 107431.

Zhu, C., Al-Dossari, M., Rezapour, S., Shateyi, S. (2023). On the exact soliton solutions and different wave structures to the modified Schrödinger's equation. *Res. Phys. 54*, 107037.

Zhu, C., Al-Dossari, M., Rezapour, S., Shateyi, S., Gunay,B. (2024). Analytical optical solutions to the nonlinear Zakharov system via logarithmic transformation. *Res. Phys. 56*, 107298.