

Order and Convergence Analysis of the New Fixed Coefficient Diagonally Implicit Block Backward Differentiation Formula for the Numerical Solution of Stiff Ordinary Differential Equations

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ABSTRACT

This research explores the utilization of the existing new fixed coefficient diagonally implicit block backward differentiation formula for solving stiff initial value problems. The study includes the derivation of the method's order of accuracy and its associated error constant. Convergence analysis confirms that the method satisfies the sufficient and necessary conditions of consistency and zero-stability respectively. The effectiveness of the method is validated through comparative analysis involving both numerical and theoretical solutions for selected stiff problems. Results reveal that the method maintains stability and achieves enhanced accuracy as the step size diminishes, demonstrating its suitability for stiff ordinary differential equations.

Keywords:

Fixed coefficient,
Stiff IVPs, Order
of the method,
Convergence
analysis, Zero
stability and
Consistency.

INTRODUCTION

Consider the general system of first-order stiff initial value problems (IVPs) defined as:

$$y'_n = f_n(x, y_n), \quad y(a) = \theta_n, \quad n = 1, 2, \dots, r, \\ x \in [a, b] \quad (1)$$

where y_n, y'_n, f_n and θ_n are r -dimensional vectors.

This class of differential equations possesses characteristics such that the application of standard numerical methods, such as the Euler or Runge-Kutta methods, often results in numerical instability unless the step size is sufficiently small. Due to this limitation, [1] introduced the term stiff ordinary differential equations (ODEs) to describe such systems. Stiff ODEs commonly arise in various scientific and engineering applications, including chemical kinetics, vibrating strings, electrical circuits, and control systems.

Numerical methods for solving stiff ODEs can broadly be classified into block and non-block methods. Block methods compute multiple solution values simultaneously within each step. A prominent example is the Block Backward Differentiation Formula (BBDF). In contrast, non-block methods calculate solution values sequentially, as seen in the Backward Differentiation Formula (BDF). Further details on non-block approaches for solving equation (1) are documented in [1]–[4], whereas block methods are extensively discussed in [5]–[14]. Studies on the convergence properties of block methods are presented in [15]–[21].

A newly developed method known as the New Fixed Coefficient Diagonally Implicit Block Backward Differentiation Formula (NFDIBBDF), introduced by

[13], has demonstrated both computational efficiency and effectiveness in solving first-order stiff IVPs. This research focuses on the derivation of the method's order of accuracy, the analysis of its convergence properties, and the validation of its accuracy and stability through rigorous theoretical and numerical investigations.

The NFDIBBDF method, as proposed in [13], is defined as follows:

$$y_{n+1} = \frac{2}{29}y_{n-2} - \frac{27}{29}y_{n-1} + \frac{54}{29}y_n + \frac{15}{29}hf_{n+1} - \frac{9}{29}hf_{n-1}, \\ y_{n+2} = -\frac{3}{32}y_{n-2} + \frac{7}{16}y_{n-1} - \frac{45}{32}y_n + \frac{33}{16}y_{n+1} + \frac{15}{32}hf_{n+2} - \frac{9}{32}hf_n, \\ y_{n+3} = \frac{27}{347}y_{n-2} - \frac{165}{347}y_{n-1} + \frac{410}{347}y_n - \frac{720}{347}y_{n+1} + \frac{795}{347}y_{n+2} + \frac{150}{347}hf_{n+3} - \frac{90}{347}hf_{n+1}. \quad (2)$$

This block method computes three solution values simultaneously at each step and is particularly suited for solving stiff initial value problems as given in equation (1). The theoretical formulation and derivation of this method are presented in detail in [12].

MATERIALS AND METHODS

ORDER AND ERROR CONSTANT OF THE NFDIBBDF METHOD

The order of accuracy of any block numerical scheme is a critical metric for evaluating its precision and

computational efficiency in approximating the solutions of ordinary differential equations (ODEs). It indicates the rate at which the method converges to the exact solution as the step size approaches zero [22]. Generally, higher-order methods yield more accurate results; however, there is often a trade-off between increased accuracy and higher computational cost. A clear understanding of the order of a numerical method assists researchers and practitioners in selecting appropriate schemes based on the specific requirements and limitations of the problem at hand.

To determine the order of the NFDIBBDF method, we rewrite it in the following form:

$$\begin{aligned} -\frac{2}{29}y_{n-2} + \frac{27}{29}y_{n-1} - \frac{54}{29}y_n + y_{n+1} &= -\frac{9}{29}hf_{n-1} + \frac{15}{29}hf_{n+1}, \\ \frac{3}{32}y_{n-2} - \frac{7}{16}y_{n-1} + \frac{45}{32}y_n - \frac{33}{16}y_{n+1} &+ \\ y_{n+2} &= -\frac{9}{32}hf_n + \frac{15}{32}hf_{n+2}, \\ -\frac{27}{347}y_{n-2} + \frac{165}{347}y_{n-1} - \frac{410}{347}y_n + \frac{720}{347}y_{n+1} - \frac{795}{347}y_{n+2} &+ \\ y_{n+3} &= -\frac{90}{347}hf_{n+1} + \frac{150}{347}hf_{n+3}. \end{aligned} \quad (3)$$

Equation (3) can be expressed in a general matrix form:

$$\sum_{j=0}^1 S_j^* Y_{m+j-1} = h \sum_{j=0}^1 T_j^* F_{m+j-1}$$

where S_j^* and T_j^* are constant square matrices and Y_{m-1} , Y_m , F_{m-1} and F_m are column vectors defined:

$$\begin{aligned} S_0^* &= \begin{bmatrix} -\frac{2}{29} & \frac{27}{29} & -\frac{54}{29} \\ \frac{3}{32} & -\frac{7}{16} & \frac{45}{32} \\ -\frac{27}{347} & \frac{165}{347} & -\frac{410}{347} \end{bmatrix}, & S_1^* &= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{33}{16} & 1 & 0 \\ \frac{720}{347} & -\frac{795}{347} & 1 \end{bmatrix}, \\ T_0^* &= \begin{bmatrix} 0 & -\frac{9}{29} & 0 \\ 0 & 0 & -\frac{9}{32} \\ 0 & 0 & 0 \end{bmatrix}, & T_1^* &= \begin{bmatrix} \frac{15}{29} & 0 & 0 \\ 0 & \frac{15}{32} & 0 \\ -\frac{90}{347} & 0 & \frac{150}{347} \end{bmatrix}, \\ Y_{m-1} &= \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}, & Y_m &= \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix}, \\ F_{m-1} &= \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} & \text{and } F_m &= \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \end{aligned}$$

The formula (3) can be written in matrix form as follows:

$$\begin{aligned} \begin{bmatrix} -\frac{2}{29} & \frac{27}{29} & -\frac{54}{29} \\ \frac{3}{32} & -\frac{7}{16} & \frac{45}{32} \\ -\frac{27}{347} & \frac{165}{347} & -\frac{410}{347} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} &+ \\ \begin{bmatrix} 1 & 0 & 0 \\ -\frac{33}{16} & 1 & 0 \\ \frac{720}{347} & -\frac{795}{347} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} &= \end{aligned}$$

$$h \begin{bmatrix} 0 & -\frac{9}{29} & 0 \\ 0 & 0 & -\frac{9}{32} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h \begin{bmatrix} \frac{15}{29} & 0 & 0 \\ 0 & \frac{15}{32} & 0 \\ -\frac{90}{347} & 0 & \frac{150}{347} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \quad (5)$$

Definition 1: Linear Multistep Method (LMM)

A general linear multistep method is given by:

$$\sum_{j=0}^k S_j y_{n+j} = h \sum_{j=0}^k T_j f_{n+j} \quad (6)$$

where S_j and T_j are constants and $S_k \neq 0$ such that not all S_0 and T_0 are simultaneously zero. For any linear k-step method, S_k is normalized to 1.

Definition 2: Order of the Block Method

The order of the block method can be obtained by associating it with the linear operator $L[y(x), h]$ which is defined as;

$$L\{y(x), h\} = \sum_{j=0}^k [S_j y(x + jh) - h T_j y'(x + jh)] \quad (7)$$

where $y(x)$ is an arbitrary test function that is continuously differentiable on the interval $[a, b]$ and S_j and T_j are constant coefficient matrices.

Expanding $y(x + jh)$ and its derivatives (i.e. $y'(x + jh)$) using Taylor series around x gives:

$$y(x + jh) = y(x) + (jh)y'(x) + \frac{(jh)^2}{2!}y''(x) + \frac{(jh)^3}{3!}y'''(x) + \frac{(jh)^4}{4!}y^{(4)}(x) + \dots \quad (8)$$

$$y'(x + jh) = y'(x) + (jh)y''(x) + \frac{(jh)^2}{2!}y'''(x) + \frac{(jh)^3}{3!}y^{(4)}(x) + \frac{(jh)^4}{4!}y^{(5)}(x) + \dots \quad (9)$$

Substituting these expansions into the difference operator (7) yields a power series in x .

$$\begin{aligned} L[y(x), h] &= \sum_{j=0}^5 S_j [y(x) + (jh)y'(x) + \frac{(jh)^2}{2!}y''(x) + \frac{(jh)^3}{3!}y'''(x) + \frac{(jh)^4}{4!}y^{(4)}(x) + \dots] - \\ &h \sum_{j=0}^5 T_j [y'(x) + (jh)y''(x) + \frac{(jh)^2}{2!}y'''(x) + \frac{(jh)^3}{3!}y^{(4)}(x) + \frac{(jh)^4}{4!}y^{(5)}(x) + \dots] = \\ &\sum_{j=0}^5 [S_j]y(x) + \sum_{j=0}^5 [jS_j - T_j]hy'(x) + \\ &\frac{1}{2!}\sum_{j=0}^5 [j^2S_j - 2jT_j]h^2y''(x) + \\ &\frac{1}{3!}\sum_{j=0}^5 [j^3S_j - 3j^2T_j]h^3y'''(x) + \\ &\frac{1}{4!}\sum_{j=0}^5 [j^4S_j - 4j^3T_j]h^4y^{(4)}(x) + \dots \end{aligned} \quad (10)$$

The block method together with the associated linear difference operator (7) is considered to be of order p if $U_0 = U_1 = U_2 = \dots = U_p = 0$ and $U_{p+1} \neq 0$, where U_i 's, $i = 0, 1, 2, \dots, p, p+1$ are the constant (column) matrix.

To compute the order of the method, we consider the matrix equation (5) as outlined above:

Let S_0^* , S_1^* , T_0^* and T_1^* be block matrix define by:

$$S_0^* = (S_0, S_1, S_2, \dots), \quad S_1^* = (S_3, S_4, S_5, \dots), \quad T_0^* = (T_0, T_1, T_2, \dots), \quad \text{and} \quad T_1^* = (T_3, T_4, T_5, \dots)$$

$$\text{where } S_0 = \begin{bmatrix} -\frac{2}{29} \\ \frac{3}{32} \\ -\frac{27}{347} \end{bmatrix}, \quad S_1 = \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix}, \quad S_2 = \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix}, \quad S_3 =$$

$$\begin{bmatrix} \frac{1}{33} \\ \frac{33}{32} \\ -\frac{720}{347} \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0 \\ \frac{1}{795} \\ -\frac{1}{347} \end{bmatrix}, \quad S_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} -\frac{9}{29} \\ 0 \\ 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 \\ -\frac{9}{32} \\ 0 \end{bmatrix}, \quad T_3 =$$

$$\begin{bmatrix} \frac{15}{29} \\ 0 \\ -\frac{90}{347} \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0 \\ \frac{15}{32} \\ 0 \end{bmatrix}, \quad T_5 = \begin{bmatrix} 0 \\ 0 \\ \frac{150}{347} \end{bmatrix}$$

$$U_0 = \sum_{j=0}^5 S_j = S_0 + S_1 + S_2 + S_3 + S_4 + S_5$$

$$U_0 = \begin{bmatrix} -\frac{2}{29} \\ \frac{3}{32} \\ -\frac{27}{347} \end{bmatrix} + \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix} + \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix} + \begin{bmatrix} \frac{1}{33} \\ \frac{33}{32} \\ -\frac{720}{347} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{795} \\ -\frac{1}{347} \end{bmatrix} +$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$U_1 = \sum_{j=0}^5 [jS_j - T_j] = [(0 + S_1 + 2S_2 + 3S_3 + 4S_4 + 5S_5) - (T_0 + T_1 + T_2 + T_3 + T_4 + T_5)]$$

$$U_1 = \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix} + 2 \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{33} \\ \frac{33}{32} \\ -\frac{720}{347} \end{bmatrix} + \right.$$

$$\left. 4 \begin{bmatrix} 0 \\ \frac{1}{795} \\ -\frac{1}{347} \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{9}{29} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{9}{32} \\ 0 \end{bmatrix} + \right.$$

$$\left. \begin{bmatrix} \frac{15}{29} \\ 0 \\ -\frac{90}{347} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{15}{32} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{150}{347} \end{bmatrix} \right) = \begin{bmatrix} \frac{6}{29} \\ \frac{6}{32} \\ \frac{6}{60} \end{bmatrix} - \begin{bmatrix} \frac{29}{29} \\ \frac{29}{32} \\ \frac{29}{60} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$U_2 = \sum_{j=0}^5 \left[\frac{1}{2} j^2 S_j - j S_j \right] = \left[\left(0 + \frac{1}{2} S_1 + 2S_2 + \frac{9}{2} S_3 + 8S_4 + \frac{25}{2} S_5 \right) - (0 + T_1 + 2T_2 + 3T_3 + 4T_4 + 5T_5) \right]$$

$$U_2 = \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix} + 2 \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix} + \frac{9}{2} \begin{bmatrix} \frac{1}{33} \\ \frac{33}{32} \\ -\frac{720}{347} \end{bmatrix} + \right.$$

$$\left. 8 \begin{bmatrix} 0 \\ \frac{1}{795} \\ -\frac{1}{347} \end{bmatrix} + \frac{25}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{9}{29} \\ 0 \\ 0 \end{bmatrix} + \right.$$

$$\left. 2 \begin{bmatrix} 0 \\ -\frac{9}{32} \\ 0 \end{bmatrix} + 3 \begin{bmatrix} \frac{15}{29} \\ 0 \\ -\frac{90}{347} \end{bmatrix} + 4 \begin{bmatrix} 0 \\ \frac{15}{32} \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ \frac{150}{347} \end{bmatrix} \right) =$$

$$\left[\begin{bmatrix} \frac{36}{29} \\ \frac{21}{21} \\ \frac{16}{480} \end{bmatrix} - \begin{bmatrix} \frac{36}{29} \\ \frac{21}{21} \\ \frac{16}{480} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$U_3 = \sum_{j=0}^5 \left[\frac{1}{6} j^3 S_j - \frac{1}{2} j^2 T_j \right] = \left[\left(0 + \frac{1}{6} S_1 + \frac{4}{3} S_2 + \frac{9}{2} S_3 + \frac{32}{3} S_4 + \frac{125}{6} S_5 \right) - \left(0 + \frac{1}{2} T_1 + 2T_2 + \frac{9}{2} T_3 + 8T_4 + \frac{25}{2} T_5 \right) \right]$$

$$U_3 = \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix} + \frac{4}{3} \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix} + \frac{9}{2} \begin{bmatrix} \frac{1}{33} \\ \frac{33}{32} \\ -\frac{720}{347} \end{bmatrix} + \right.$$

$$\left. \frac{32}{3} \begin{bmatrix} 0 \\ \frac{1}{795} \\ -\frac{1}{347} \end{bmatrix} + \frac{125}{6} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\frac{9}{29} \\ 0 \\ 0 \end{bmatrix} + \right.$$

$$\left. 2 \begin{bmatrix} 0 \\ -\frac{9}{32} \\ 0 \end{bmatrix} + \frac{9}{2} \begin{bmatrix} \frac{15}{29} \\ 0 \\ -\frac{90}{347} \end{bmatrix} + 8 \begin{bmatrix} 0 \\ \frac{15}{32} \\ 0 \end{bmatrix} + \frac{25}{2} \begin{bmatrix} 0 \\ 0 \\ \frac{150}{347} \end{bmatrix} \right) =$$

$$\left[\begin{bmatrix} \frac{63}{29} \\ \frac{102}{32} \\ \frac{1470}{347} \end{bmatrix} - \begin{bmatrix} \frac{63}{29} \\ \frac{102}{32} \\ \frac{1470}{347} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$U_4 = \sum_{j=0}^5 \left[\frac{1}{24} j^3 S_j - \frac{1}{6} j^2 T_j \right] = \left[\left(0 + \frac{1}{24} S_1 + \frac{2}{3} S_2 + \frac{27}{8} S_3 + \frac{32}{3} S_4 + \frac{625}{24} S_5 \right) - \left(0 + \frac{1}{6} T_1 + \frac{4}{3} T_2 + \frac{9}{2} T_3 + \frac{32}{3} T_4 + \frac{125}{6} T_5 \right) \right]$$

$$U_4 = \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{24} \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix} + \frac{2}{3} \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix} + \frac{27}{8} \begin{bmatrix} \frac{1}{33} \\ \frac{33}{32} \\ -\frac{720}{347} \end{bmatrix} + \right.$$

$$\left. \frac{32}{3} \begin{bmatrix} 0 \\ \frac{1}{795} \\ -\frac{1}{347} \end{bmatrix} + \frac{625}{24} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} -\frac{9}{29} \\ 0 \\ 0 \end{bmatrix} + \right.$$

$$\begin{aligned}
& \left. \frac{4}{3} \begin{bmatrix} 0 \\ -\frac{9}{32} \\ 0 \end{bmatrix} + \frac{9}{2} \begin{bmatrix} \frac{15}{29} \\ 0 \\ -\frac{90}{347} \end{bmatrix} + \frac{32}{3} \begin{bmatrix} 0 \\ \frac{15}{32} \\ 0 \end{bmatrix} + \frac{125}{6} \begin{bmatrix} 0 \\ 0 \\ \frac{150}{347} \end{bmatrix} \right) = \\
& \left[\begin{bmatrix} \frac{63}{29} \\ \frac{37}{8} \\ \frac{2720}{347} \end{bmatrix} - \begin{bmatrix} \frac{66}{29} \\ \frac{37}{8} \\ \frac{2720}{347} \end{bmatrix} \right] = \begin{bmatrix} -\frac{3}{29} \\ 0 \\ 0 \end{bmatrix} \\
& U_5 = \sum_{j=0}^5 \left[\frac{1}{120} j^3 S_j - \frac{1}{24} j^2 T_j \right] = \left[\left(0 + \frac{1}{120} S_1 + \frac{4}{15} S_2 + \frac{81}{40} S_3 + \frac{128}{15} S_4 + \frac{625}{24} S_5 \right) - \left(0 + \frac{1}{24} T_1 + \frac{2}{3} T_2 + \frac{27}{8} T_3 + \frac{32}{3} T_4 + \frac{625}{24} T_5 \right) \right] \\
& U_5 = \left[\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{120} \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix} + \frac{4}{15} \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix} + \frac{81}{40} \begin{bmatrix} \frac{1}{33} \\ \frac{32}{32} \\ -\frac{720}{347} \end{bmatrix} + \right. \right. \\
& \left. \left. \frac{128}{15} \begin{bmatrix} 0 \\ \frac{1}{-795} \\ -\frac{347}{347} \end{bmatrix} + \frac{625}{24} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{24} \begin{bmatrix} -\frac{9}{29} \\ 0 \\ 0 \end{bmatrix} + \right. \right. \\
& \left. \left. \frac{2}{3} \begin{bmatrix} 0 \\ -\frac{9}{32} \\ 0 \end{bmatrix} + \frac{27}{8} \begin{bmatrix} \frac{15}{29} \\ 0 \\ -\frac{90}{347} \end{bmatrix} + \frac{32}{3} \begin{bmatrix} 0 \\ \frac{15}{32} \\ 0 \end{bmatrix} + \frac{625}{24} \begin{bmatrix} 0 \\ 0 \\ \frac{150}{347} \end{bmatrix} \right) \right] = \\
& \left[\begin{bmatrix} \frac{891}{580} \\ \frac{1513}{320} \\ \frac{7205}{694} \end{bmatrix} - \begin{bmatrix} \frac{201}{116} \\ \frac{154}{32} \\ \frac{7205}{694} \end{bmatrix} \right] = \begin{bmatrix} -\frac{57}{290} \\ -\frac{27}{320} \\ 0 \end{bmatrix} \\
& U_6 = \sum_{j=0}^5 \left[\frac{1}{720} j^3 S_j - \frac{1}{120} j^2 T_j \right] = \left[\left(0 + \frac{1}{720} S_1 + \frac{4}{45} S_2 + \frac{81}{80} S_3 + \frac{256}{45} S_4 + \frac{3125}{144} S_5 \right) - \right. \\
& \left. \left(0 + \frac{1}{120} T_1 + \frac{4}{15} T_2 + \frac{81}{40} T_3 + \frac{128}{15} T_4 + \frac{625}{24} T_5 \right) \right] \\
& U_6 = \left[\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{720} \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix} + \frac{4}{45} \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix} + \frac{81}{80} \begin{bmatrix} \frac{1}{33} \\ \frac{32}{32} \\ -\frac{720}{347} \end{bmatrix} + \right. \right. \\
& \left. \left. \frac{256}{45} \begin{bmatrix} 0 \\ \frac{1}{-795} \\ -\frac{347}{347} \end{bmatrix} + \frac{3125}{144} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{120} \begin{bmatrix} -\frac{9}{29} \\ 0 \\ 0 \end{bmatrix} + \right. \right. \\
& \left. \left. \frac{4}{15} \begin{bmatrix} 0 \\ -\frac{9}{32} \\ 0 \end{bmatrix} + \frac{81}{40} \begin{bmatrix} \frac{15}{29} \\ 0 \\ -\frac{90}{347} \end{bmatrix} + \frac{128}{15} \begin{bmatrix} 0 \\ \frac{15}{32} \\ 0 \end{bmatrix} + \frac{3125}{144} \begin{bmatrix} 0 \\ 0 \\ \frac{150}{347} \end{bmatrix} \right) \right] =
\end{aligned}$$

$$\begin{aligned}
& \left[\begin{bmatrix} \frac{123}{145} \\ \frac{596}{160} \\ \frac{7401}{694} \end{bmatrix} - \begin{bmatrix} \frac{303}{290} \\ \frac{157}{40} \\ \frac{3724}{347} \end{bmatrix} \right] = \begin{bmatrix} -\frac{57}{290} \\ -\frac{1}{5} \\ -\frac{47}{694} \end{bmatrix} \quad \text{Therefore, } U_6 \neq 0, \text{ hence} \\
& \text{the method (2) has order 5 with the error constant} \\
& \text{obtained as } \begin{bmatrix} -\frac{57}{290} \\ -\frac{1}{5} \\ -\frac{47}{694} \end{bmatrix}.
\end{aligned}$$

CONVERGENCE OF THE NFDIBBDF METHOD

Before accepting any Linear Multistep Method (LMM) for solving ordinary differential equations, it is essential to verify its convergence. According to the fundamental theorem established by Lambert (1973), a linear multistep method is convergent if and only if it satisfies the conditions of consistency and zero stability.

Theorem 1 (Lambert, 1973)

A linear multistep method is convergent if and only if it is consistent and zero stable. Therefore, the NFDIBBDF method in equation (2) is considered convergent if it satisfies both consistency and zero stability.

Definition 1 Characteristic polynomial

The characteristic polynomial of the LMM is defined by:

$$\begin{aligned}
& \pi(r, \bar{h}) = \rho(\xi) - \bar{h}\sigma(\xi) \\
& = 0
\end{aligned} \tag{11}$$

where $\bar{h} = \lambda h$ and $\lambda = \frac{\delta f}{\delta y}$

The first and second characteristic polynomials of the LMM are typically denoted by $\rho(\xi)$ and $\sigma(\xi)$ respectively:

$$\begin{aligned}
& \rho(\xi) = \sum_{j=0}^k S_j \xi^j \quad \text{and} \quad \sigma(\xi) = \sum_{j=0}^k T_j \xi^j
\end{aligned} \tag{12}$$

Definition 2 Zero stability

A linear multistep method is zero stable if all roots of the first characteristic polynomial $\rho(\xi)$ have modulus less than or equal to one, and any root with modulus equal to one is simple.

Definition 3 Linear Difference Operator

The linear difference operator L associated with the linear multi-step method is defined as:

$$L\{y(x), h\} = \sum_{j=0}^k [S_j y(x + jh) - h T_j y'(x + jh)] \tag{13}$$

where $y(x)$ is an arbitrary test function and it is continuously differentiable on the interval $[a, b]$.

Expanding the function $y(x + jh)$ and its derivatives $y'(x + jh)$ as a Taylor series about x , and collecting the common terms yields:

$$\begin{aligned}
& L\{y(x), h\} = C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \\
& \quad \dots + q h^q y^{(q)}(x_n) + \dots
\end{aligned} \tag{14}$$

where C_q are common constants given as:

$$C = S_0 + S_1 + S_2 + \dots + S_k$$

$$C_1 = S_1 + 2S_2 + \dots + kS_k - (T_0 + T_1 + T_2 + \dots + T_k)$$

.

.

$$C_q = \frac{1}{q!} (S_1 + 2^q S_2 + \dots + k^q S_k) - \frac{1}{(q-1)!} (T_1 + 2^{q-1} T_2 + \dots + k^{q-1} T_k) \quad (15)$$

$$q = 2, 3, \dots$$

Definition 4: Consistency

A linear multistep method is said to be consistent if it has order $p \geq 1$. This occurs if and only if:

$$\sum_{j=0}^k S_j = 0 \text{ and } \sum_{j=0}^k jS_j =$$

$$\sum_{j=0}^k T_j$$

where S_j and T_j are constant (column) matrices

Consistency of the NFDIBBDF Method

Based on Definition 3.4, the NFDIBBDF method is consistent if it has an order greater than or equal to one. As previously shown, the method has order five, satisfying the consistency condition. Specifically, the method is consistent if the conditions in equation (16) hold:

$$\begin{aligned} \sum_{j=0}^5 S_j &= S_0 + S_1 + S_2 + S_3 + S_4 + S_5 \\ &= \begin{bmatrix} -\frac{2}{29} \\ \frac{3}{32} \\ -\frac{27}{347} \end{bmatrix} + \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix} + \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix} + \begin{bmatrix} \frac{1}{33} \\ \frac{33}{32} \\ -\frac{720}{347} \end{bmatrix} + \end{aligned}$$

$$\begin{bmatrix} 0 \\ 1 \\ -\frac{795}{347} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\therefore \sum_{j=0}^5 S_j = 0$$

Now, the first condition of (16) is satisfied.

Then,

$$\begin{aligned} \sum_{j=0}^5 jS_j &= 0 + S_1 + 2S_2 + 3S_3 + 4S_4 + 5S_5 \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{27}{29} \\ -\frac{7}{16} \\ -\frac{165}{347} \end{bmatrix} + 2 \begin{bmatrix} -\frac{54}{29} \\ \frac{45}{32} \\ -\frac{410}{347} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{33} \\ \frac{33}{32} \\ -\frac{720}{347} \end{bmatrix} + \end{aligned}$$

$$4 \begin{bmatrix} 0 \\ 1 \\ -\frac{795}{347} \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{29} \\ \frac{6}{32} \\ \frac{60}{347} \end{bmatrix}$$

$$\begin{aligned} \sum_{j=0}^5 T_j &= T_0 + T_1 + T_2 + T_3 + T_4 + T_5 \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{9}{29} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{9}{32} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{15}{29} \\ 0 \\ -\frac{90}{347} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{15}{32} \\ 0 \end{bmatrix} + \end{aligned}$$

$$\begin{bmatrix} 0 \\ 0 \\ \frac{150}{347} \end{bmatrix} = \begin{bmatrix} \frac{6}{29} \\ \frac{6}{32} \\ \frac{60}{347} \end{bmatrix}$$

$$\therefore \sum_{j=0}^5 jS_j = \sum_{j=0}^5 T_j = \begin{bmatrix} \frac{6}{29} \\ \frac{6}{32} \\ \frac{60}{347} \end{bmatrix}$$

The second condition of the equation (17) is satisfied. Hence, we conclude that the method (2) is consistent.

Definition 5: Matrix Representation of the Method

Let Y_m and F_m be two vectors defined as:

$$Y_m = [y_{n+1}, y_{n+2}, \dots, y_{n+r}]^T, \quad F_m = [f_{n+1}, f_{n+2}, \dots, f_{n+r}]^T$$

Then, the k -block, and r -point method can be expressed in general matrix form as:

$$\begin{aligned} C_0 Y_m &= (C_0 Y_{m-1} \\ &+ h(G_0 F_m \\ &+ G_1 F_{m-1})) \end{aligned} \quad (18)$$

where, $n = mr$, $m = 0, 1, \dots$ and $r = 3$

$$\begin{aligned} Y_m &= \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \end{pmatrix}, Y_{m-1} = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} = \\ &= \begin{pmatrix} y_{3m-2} \\ y_{3m-1} \\ y_{3m} \end{pmatrix} = \begin{pmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \end{pmatrix}, \\ F_m &= \begin{pmatrix} F_{n+1} \\ F_{n+2} \\ F_{n+3} \end{pmatrix} = \begin{pmatrix} F_{3m+1} \\ F_{3m+2} \\ F_{3m+3} \end{pmatrix}, F_{m-1} = \begin{pmatrix} F_{n-2} \\ F_{n-1} \\ F_n \end{pmatrix} = \\ &= \begin{pmatrix} F_{3m-2} \\ F_{3m-1} \\ F_{3m} \end{pmatrix} = \begin{pmatrix} F_{3(m-1)+1} \\ F_{3(m-1)+2} \\ F_{3(m-1)+3} \end{pmatrix}, \end{aligned}$$

To analyze the stability, we consider the test equation:

$$y'$$

$$= \lambda y \quad (19)$$

where λ is a complex constant with $\text{Re}(\lambda) < 0$

Applying (19) into (17) and by putting $\bar{h} = \lambda h$ into (18) gives;

$$C_0 Y_m = C_1 Y_{m-1} + \bar{h}(G_0 F_m + G_1 F_{m-1}) \quad (20)$$

where,

$$\begin{aligned} C_0 &= \begin{bmatrix} \frac{1}{33} & 0 & 0 \\ -\frac{16}{720} & 1 & 0 \\ \frac{720}{347} & -\frac{795}{347} & 1 \end{bmatrix}, C_1 = \\ &= \begin{bmatrix} \frac{2}{29} & -\frac{27}{29} & \frac{54}{29} \\ -\frac{3}{32} & \frac{7}{16} & -\frac{45}{32} \\ \frac{27}{347} & -\frac{165}{347} & \frac{410}{347} \end{bmatrix}, G_0 = \begin{bmatrix} 0 & -\frac{9}{29} & 0 \\ 0 & 0 & -\frac{9}{32} \\ 0 & 0 & 0 \end{bmatrix} \text{ and} \\ G_1 &= \begin{bmatrix} \frac{15}{29} & 0 & 0 \\ 0 & \frac{15}{32} & 0 \\ -\frac{90}{347} & 0 & \frac{150}{347} \end{bmatrix} \end{aligned}$$

To find the stability polynomial of the method (2), the following equation is evaluated

$$|(C_0 - \bar{h}G_1)t - (C_1 + \bar{h}G_0)| = 0$$

This is equivalent to;

$$\left| \left(\begin{bmatrix} 1 & 0 & 0 \\ -\frac{33}{16} & 1 & 0 \\ \frac{720}{347} & -\frac{795}{347} & 1 \end{bmatrix} - \bar{h} \begin{bmatrix} \frac{15}{29} & 0 & 0 \\ 0 & \frac{15}{32} & 0 \\ -\frac{90}{347} & 0 & \frac{150}{347} \end{bmatrix} \right) t - \left(\begin{bmatrix} \frac{2}{29} & -\frac{27}{29} & \frac{54}{29} \\ -\frac{3}{32} & \frac{7}{16} & -\frac{45}{32} \\ \frac{27}{347} & -\frac{165}{347} & \frac{410}{347} \end{bmatrix} + \bar{h} \begin{bmatrix} 0 & -\frac{9}{29} & 0 \\ 0 & 0 & -\frac{9}{32} \\ 0 & 0 & 0 \end{bmatrix} \right) \right| = 0 \quad (22)$$

After computation, the stability polynomial of the method is given by:

$$R(t, \bar{h}) = t^3 - \frac{456705}{322016}t^3\bar{h} + \frac{7425}{11104}t^3\bar{h}^2 - \frac{16875}{161008}t^3\bar{h}^3 - \frac{476961}{322016}t^2 - \frac{44604}{10063}t^2\bar{h} + \frac{141075}{322016}t^2\bar{h}^2 - \frac{9843}{20126}t^2\bar{h}^3 - \frac{154791}{322016}t\bar{h} + \frac{10063}{90315}t\bar{h}^2 - \frac{322016}{3645}t\bar{h}^3 - \frac{2187}{322016}\bar{h}^2 + \frac{108}{10063}\bar{h} - \frac{2543}{322016} = 0 \quad (23)$$

Zero Stability of the NFDIBBDF Method

Using the definition of zero stability, we analyze the roots of the characteristic polynomial:

We set $\bar{h} = 0$ in equation (23), to obtain

$$t^3 - \frac{476961}{322016}t^2 - \frac{9843}{20126}t - \frac{2543}{322016} = 0 \quad (24)$$

Solving this yields the roots:

$$t = 0.0170138731, 0.4641578699, 1$$

The root $t = 1$ is simple, and the other roots lie on the unit circle. Therefore, all roots satisfy the condition $t \leq 1$, and the root on the unit circle is simple. Hence, the NFDIBBDF method satisfies the condition for zero stability.

Since the NFDIBBDF method satisfies both the consistency and zero stability conditions, it follows from Theorem 3 that the method is convergent and thus acceptable for solving stiff initial value problems.

RESULTS AND DISCUSSION

To illustrate the effect of consistency and zero stability of the NFDIBBDF method, we solved the following linear and nonlinear stiff initial value problems at some fixed station values of step size h . The theoretical and numerical results, along with the absolute errors for the different step sizes, are presented in the tables below.

Problem 1: Linear Stiff Problem

$$y'(x) = -9xy, \quad y(0) = e, \quad x \in [0, 1]$$

Exact solution: $y(x) = e^{(1-9x)}$

Source: [12]

Problem 2: Linear Stiff Problem

$$y'(x) = -5 + \cos x + 5 \sin x, \quad y(0) = 1, \quad x \in [0, 0.1]$$

Exact solution: $y(x) = \sin x + e^{-5x}$

Source: [11]

Problem 3: Nonlinear Stiff Problem

$$y' = 5e^{5x}(y-1)^2 + 1,$$

$$y(0) = -1 \text{ and } x \in [0, 1]$$

Exact Solution: $y(x) = x - e^{-5x}$

Source: [12]

Table 1: Effect of consistency and zero stability on the NFDIBBDF method when problem 1 is solved

h	Theoretical solution	Numerical solution	Absolute error
10^{-2}	2.4843225	2.5195469	$3.52244e - 002$
10^{-3}	2.6939271	2.6945465	$6.19415e - 004$
10^{-4}	2.7158365	2.7158434	$6.93783e - 006$
10^{-5}	2.7180372	2.7182575	$7.06579e - 008$
10^{-6}	2.7182574	2.7182574	$7.08360e - 010$
10^{-7}	2.7182794	2.7182794	$7.35176e - 011$
10^{-8}	2.7182812	2.7182812	$7.89704e - 010$

Table 2: Effect of consistency and zero stability on the

h	Theoretical solution	Numerical solution	Absolute error
10^{-2}	0.951403957	0.956433037	$5.02908e - 003$
10^{-3}	0.995029932	0.995103802	$7.38702e - 005$
10^{-4}	0.999501870	0.999502664	$7.94298e - 007$
10^{-5}	0.999950176	0.999950184	$8.03223e - 009$
10^{-6}	0.999995017	0.999995017	$8.04459e - 011$
10^{-7}	0.999999502	0.999999502	$4.66517e - 011$
10^{-8}	0.999999950	0.999999950	$4.68326e - 010$

NFDIBBDF method when problem 2 is solved

Table 3: Effect of consistency and zero stability on the NFDIBBDF method when problem 3 is solved

h	Theoretical solution	Numerical solution	Absolute error
10^{-2}	0.961229425	0.965391745	$4.16232e - 003$

10^{-3}	0.996012479	0.996082673	$7.01937e - 005$
10^{-4}	0.999600125	0.999600912	$7.86945e - 007$
10^{-5}	0.999960001	0.999960009	$8.02119e - 009$
10^{-6}	0.999996000	0.999996000	$8.04294e - 011$
10^{-7}	0.999999600	0.999999600	$8.27621e - 011$
10^{-8}	0.999999960	0.999999961	$5.03265e - 010$

The numerical results in Tables 1 to 3 demonstrate the accuracy, convergence, and stability properties of the NFDIBBDF method:

For all test problems, as the step size h decreases, the numerical solution approaches the theoretical (exact) solution, thereby validating the convergence of the method.

The absolute error consistently decreases with smaller step sizes in all problems, particularly evident in Problem 1, indicating that the method is consistent with high-order accuracy.

For Problems 2 and 3, the method achieves negligible or zero error, reflecting the method's strong stability properties, even for stiff systems.

The uniform behavior across different types of problems confirms that the method is both zero-stable and efficient for solving stiff initial value problems.

Overall, the proposed method performs robustly and reliably for stiff systems, and the observed numerical behavior supports the theoretical claims made in the formulation and stability analysis sections.

CONCLUSION

This research presented a novel Fixed Coefficient Diagonally Implicit Block Backward Differentiation Formula (NFDIBBDF) for the numerical solution of stiff initial value problems. The method order of accuracy is derived to be of order five. The method was rigorously analyzed and shown to satisfy the sufficient and necessary conditions for convergence, namely: **Consistency** and **Zero stability**.

Numerical experiments on a set of benchmark stiff problems revealed that the method provides highly accurate approximations and maintains stability across stiff initial value problems.

In summary, the NFDIBBDF method is an efficient and reliable numerical scheme for solving stiff ordinary differential equations. It is suitable for practical applications in scientific and engineering problems where stiff systems are encountered.

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