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A New Weibull-Gamma Distribution: Theory, Estimation and Applications

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ABSTRACT
In the fields of reliability engineering, survival analysis, and lifetime data modeling, accurately representing the failure times and life durations of systems, components, and organisms is a central concern. Traditional lifetime distributions—such as the exponential, Weibull, and gamma distributions—have been widely used due to their mathematical tractability and interpretability. However, these classical models often struggle to capture the complexity of real-world data, particularly when hazard rate behaviors vary, including increasing, decreasing, bathtub-shaped, or unimodal patterns. Although the Weibull distribution is popular for its flexibility, it may not sufficiently model datasets where the hazard function deviates from its typical monotonic form. Similarly, while the gamma distribution is effective in many stochastic and queuing contexts, it lacks the versatility to represent certain tail behaviors and multimodal characteristics observed in practice. To address these limitations, statisticians have developed hybrid and compound distributions that merge features from multiple distributions, enhancing both flexibility and applicability. One such development is the Weibull-Gamma distribution, derived by mixing the Weibull and Gamma distributions. The first four moments about the origin, as well as the mean, were calculated for this new distribution. Derived expressions also include the coefficient of variation, skewness, kurtosis, and index of dispersion. In addition, the moment-generating function, characteristic function, and Laplace transform were established. Key reliability functions—such as the survival function, hazard rate function, and mean residual life—were also derived. Parameter estimation was carried out using the Maximum Likelihood Estimation (MLE) method. The goodness-of-fit of the proposed distribution was evaluated against several existing related models using criteria such as the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICC), and Bayesian Information Criterion (BIC). These comparisons were based on real-world datasets. The results demonstrated that the Weibull-Gamma distribution outperformed the competing models, making it a promising alternative for modeling real-life lifetime data.

Keywords:
Probability,
Gamma,
Weibull,
Goodness of fit,
maximum likelihood method.

INTRODUCTION

The Weibull and Gamma distributions are widely recognized for their applications in reliability analysis, lifetime modeling, and event-based data analysis. The Weibull distribution, with its shape and scale parameters, is particularly useful for modeling survival and failure times (Umar & Yahya, 2021). Similarly, the Gamma distribution, often applied to model waiting times or life durations, is governed by its shape and rate parameters. These two distributions, when combined into the Weibull-Gamma (WG) distribution, provide a powerful framework for modeling data where events are influenced by both multiplicative degradation and additive randomness, making it especially useful in complex lifetime data modeling (Umar, Jimoh, & Yahya, 2019). The Weibull-Gamma distribution is notable for its ability to model varying hazard functions, making it particularly well-suited for data that exhibits both increasing and decreasing failure rates. Its versatility has been demonstrated in fields such as healthcare, engineering, and economics (Yahya & Umar, 2024). To achieve this, statisticians have developed and refined various

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lifetime distributions (Amiru et al, 2025; Manu et al, 2023; Olalekan et al, 2021) that can accommodate the diverse patterns of hazard functions observed in real-world phenomena. The most general form of the gamma distribution is the three-parameter Generalized Gamma (GG) distribution (Stacy, 1962). The distribution is suitable for modeling data having different types of hazard rate functions; increasing, decreasing, bathtub shaped and unimodal, which makes it particularly useful for estimating individual hazard functions. The GG distribution has been used in several research areas such as engineering, hydrology and survival analysis (Shanker & Shukla, 2017). Gamma distribution is very versatile and gives useful presentations of many physical situations. It is perhaps the most applied statistical distribution in analysis of reliability (Nadarajah, 2008). The GG model, having Weibull, Gamma and Exponential distributions as special sub-models among others, is a very popular distribution for modeling lifetime data and for modeling phenomenon with monotone failure rates. It plays a very important role in statistical inferential problems. A generalization of the Generalized Gamma (GGG) distribution, which includes the three-parameter generalized gamma (GG) distribution, two-parameter Weibull and gamma distributions, and exponential distribution, has been suggested and investigated by Shanker & Shukla (2019). The behavior of the hazard rate function of the distribution has been discussed. The estimation of the parameters of the distribution has been explained using the method of maximum likelihood. The goodness-of-fit of the distribution has been discussed and the fit was quite satisfactory over GG, Gamma, Weibull, and Exponential distributions.

The Lindley distribution was originally introduced by Lindley (1958) in the context of Fiducial and Bayesian Statistics. In the context of reliability studies, Ghitany, Atieh & Nadarajah (2008) among others studied in great detail the Lindley distribution; a detailed study on its various properties, parameter estimation, and application showing its superiority over Gamma, Exponential and Weibull distributions. This Lindley distribution has been modified, extended and generalized along with its applications to different fields of knowledge (Abouammoh, Alshangiti&Ragab, 2015; Alkarni, 2015; Bhati, Malik &Vaman, 2015; Ghitany, Al-Mutairi, Balakrishnan& Al-Enezi, 2013; Parai, Liyanage, & Oluyede, 2015; Sharma, Singh, Singh & Agiwal, 2015; Wang, 2013; Warahena-Liyanage & Pararai, 2014, among others). This is because there are situations in which the original Lindley distribution may not be suitable from a theoretical or applied point of view (Ghitany, et al., 2013). So, to obtain a more flexible density function than Lindley.

In the realm of reliability engineering, survival analysis, and lifetime data modeling, the accurate representation of failure times and life durations of systems, components, and organisms is central concern. Traditional lifetime distributions such as the exponential, Weibull, and gamma models have been extensively used due to their mathematical tractability and interpretability. However, these classical models often fall short when dealing with complex data that exhibit various types of hazard rate behaviours such as increasing, decreasing, bathrub-shaped, or unimodal patterns. The Weibull distribution, despite its popularity and flexibility, may not adequately model data sets where the shape of the hazard function deviates from its typical monotonic behaviour. Similarly, while the gamma distribution provides a good fit in many stochastic and queuing processes, it lacks the versatility to capture certain tail behaviors and multimodal characteristics observed in real-life data. To bridge this gap, statisticians have proposed hybrid and compounded distributions that combine the properties of two or more distributions to enhance flexibility and applicability. The Weibull-Gamma distribution, a newly proposed compound lifetime distribution, seeks to address the limitations of existing models by integrating the strengths of the Weibull and gamma distributions. This distribution introduces additional parameters that allow it to better model various hazard rate shapes and improve goodness of fit for empirical data. Despite its promising theoretical potential, this distribution is still in its early stages of development, and several critical questions remain unanswered: (1) what are the theoretical properties of the Weibull-Gamma distribution (e.g., moments, entropy, hazard rate function)? (2) How does this distribution perform compared to existing lifetime models in terms of flexibility and fit? (3) what estimation techniques can be effectively applied to this distribution, and how efficient are these methods? (4) can the Weibull-Gamma distribution be effectively used in real-world applications used as biomedical studies, industrial reliability data, or actuarial science?. Addressing these question is crucial for validating the Weibull-Gamma distribution as a reliable tool in statistical modeling. Therefore, this research aims to thoroughly, investigate the theoretical foundation, mathematical properties, estimated and applicability of the new lifetime model, the Weibull-Gamma distribution, to determine its viability and usefulness in practice.

MATERIALS AND METHODS

The Weibull distribution is defined by its probability density function as;

$$f_1(x;\beta,\theta) = \beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^{\beta}\right); x > 0, \beta > 0, \theta > 0 \qquad (1)$$

The Gamma distribution (Stacy, 1962; Artemiou, 2009; Shanker & Shukla, 2019) is defined by its probability density function as;

$$f_2(x;\alpha,\theta)=\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)};x>0,\alpha>0,\theta>0 \quad (2)$$

The New Weibull-Gamma Distribution
The mixture of both the $f_1(x;\beta,\theta)$ and $f_2(x;\alpha,\theta)$ in (1) and (2) with the mixing parameter π yielded another density function of the form;

$$f(x;\beta,\alpha,\theta)=\pi.f_1(x;\beta,\theta)+(1-\pi)f_2(x;\alpha,\theta) \quad (3)$$

The mixing parameter π is defined by

$$\pi=\frac{\theta}{\theta+\Gamma(\alpha)}\ with\ (1-\pi)=\frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}\ such\ that\ \pi+(1-\pi)=1$$

Thus, substituting the density functions $f_1(x;\beta,\theta)$ given by (1) and $f_2(x;\alpha,\theta)$ given by (2) into (3), we have new density function given by

$$f(x;\beta,\alpha,\theta)=\frac{\theta}{\theta+\Gamma(\alpha)}.\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^{\beta}\right)+\frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}.\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)} \quad (4)$$

After the little algebra, we have its final form as;

$$f(x;\beta,\alpha,\theta)=\frac{\theta\left[\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^{\beta}\right)+(\theta x)^{\alpha-1}\exp(-\theta x)\right]}{\theta+\Gamma(\alpha)} \quad (5)$$

The density function in (5) is called the new Weibull-Gamma (WG) density function.

To confirm the Weibull-Gamma Distribution is a valid probability density
To verify that the new Weibull-Gamma (WG) distribution in (5) satisfies the properties of a true probability density function (pdf), we must check two key properties of valid probability density function:
(1) **Non-negativity:**$f(x;\beta,\alpha,\theta)\geq 0\ for\ all\ x>0$
(2) **Normalization**:we need to verify that:

$$\int_0^{\infty} f(x;\beta,\alpha,\theta)\,dx=1.\ Hence, it\ follows\ that: substituting\ f(x;\beta,\alpha,\theta)$$

$$\int_0^{\infty} f(x;\beta,\alpha,\theta)\,dx=\int_0^{\infty}\frac{\theta\left[\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^{\beta}\right)+(\theta x)^{\alpha-1}\exp(-\theta x)\right]}{\theta+\Gamma(\alpha)}dx$$

Since integration is linear, we can split the integral:

$$\Rightarrow\frac{\theta}{\theta+\Gamma(\alpha)}\left[\beta\theta\int_0^{\infty}(\theta x)^{\beta-1}\exp\left(-(\theta x)^{\beta}\right)dx+\int_0^{\infty}(\theta x)^{\alpha-1}\exp(-\theta x)\,dx\right] \quad (6)$$

First component: Weibull Distribution integral:

$$\beta\theta\int_0^{\infty}(\theta x)^{\beta-1}\exp\left(-(\theta x)^{\beta}\right)dx\left(substituting\ u=(\theta x)^{\beta}\Rightarrow(\theta x)=u^{\frac{1}{\beta}}\Rightarrow dx=\frac{1}{\beta\theta}u^{\frac{1}{\beta}-1}du\right)$$

$$\Rightarrow\beta\theta\int_0^{\infty}\left(u^{\frac{1}{\beta}}\right)^{\beta-1}\exp(-u)\frac{1}{\beta\theta}u^{\frac{1}{\beta}-1}du=\int_0^{\infty}\exp(-u)\,du=1 \quad (7)$$

Second component: Gamma Distribution integral:

$$\int_0^{\infty}(\theta x)^{\alpha-1}\exp(-\theta x)\,dx\ \left(substituting\ u=(\theta x)\Rightarrow x=\frac{u}{\theta}\Rightarrow dx=\frac{du}{\theta}\right)$$

$$\Rightarrow\int_0^{\infty}(u)^{\alpha-1}\exp(-u).\frac{du}{\theta}=\frac{1}{\theta}\int_0^{\infty}(u)^{\alpha-1}\exp(-u)\,du=\frac{\Gamma(\alpha)}{\theta} \quad (8)$$

Putting (7) and (8) in (6), we get:

$$\int_0^{\infty}f(x;\beta,\alpha,\theta)\,dx=\frac{\theta}{\theta+\Gamma(\alpha)}\left[1+\frac{\Gamma(\alpha)}{\theta}\right]=\frac{\theta}{\theta+\Gamma(\alpha)}.\frac{\theta+\Gamma(\alpha)}{\theta}=1 \quad (9)$$

Conclusion: since $f(x;\beta,\alpha,\theta)\geq 0\ for\ all\ x\geq 0$ and integrates to 1, the WG density function, $f(x;\beta,\alpha,\theta)$ in (5) is valid.

The Cumulative Distribution Function of the new Weibull-Gamma (WG) Distribution
The cumulative Distribution function (CDF) of a new WG density function is given by:

$$F(x;\beta,\alpha,\theta)=\pi.F_1(x;\beta,\theta)+(1-\pi)F_2(\alpha,\theta) \quad (10)$$

Step 1: Compute $F_1(x;\beta,\theta)$ [CDF of $f_1(x;\beta,\theta)$is density function of Weibull distribution]
The cumulative distribution function for a Gamma distribution is given by

$$F_1(x;\beta,\theta)=1-\exp\left(-(\theta x)^{\beta}\right) \quad (11)$$

Step 2: Compute $F_2(x;\alpha,\theta)$ [CDF of $f_2(x;\alpha,\theta)$is density function of Gamma distribution]
The cumulative distribution function for a Gamma distribution is given by the lower incomplete gamma function

$$F_2(x;\alpha,\theta)=\frac{\gamma(\alpha,\theta x)}{\Gamma(\alpha)} \quad (12)$$

where $\gamma(\alpha, \theta x)$ is the lower incomplete gamma function:

$$\gamma(\alpha, \theta x) = \int_0^{\theta x} t^{\alpha-1} \exp(-t)\, dt \qquad (13)$$

Substituting for $F_1(x; \beta, \theta)$ and $F_2(x; , \alpha, \theta)$ in (10), we get:

$$F(x; \beta, \alpha, \theta) = \frac{\theta\left(1 - \exp\left(-(\theta x)^{\beta}\right)\right)}{\theta + \Gamma(\alpha)} + \frac{\Gamma(\alpha)}{\theta + \Gamma(\alpha)} \cdot \frac{\gamma(\alpha, \theta x)}{\Gamma(\alpha)} \qquad (14)$$

After simplification, the final expression for the cumulative distribution function is

$$F(x; \beta, \alpha, \theta) = \frac{\theta\left(1 - \exp\left(-(\theta x)^{\beta}\right)\right) + \gamma(\alpha, \theta x)}{\theta + \Gamma(\alpha)} \qquad (15)$$

The Graphs of Probability Density and Cumulative Distribution Functions of the new Weibull-Gamma (WG) Distribution

The density plot of the proposed model WGD at various values of $\alpha, \theta, and\ \beta$ is shown in figure 1. It can be observed that:

1. Effect of α (Shape Parameter):
   - When α increases, it generally shifts the peak of the PDF to the right and increases the spread of the distribution. This means the distribution becomes more spread out (e.g., more likely to take larger values).
   - For smaller values of α, the PDF will peak sharply and decay more quickly.
   - In the **CDF**, a higher α will cause the cumulative probability to rise more gradually over the range of X.
2. Effect of $\theta\ and\ \beta$ (Scale Parameters):
   - Larger values of $\theta\ and\ \beta$ spread out the distribution and decrease the peak of the PDF. It also shifts the distribution rightward, making larger values of X more probable.
   - In the CDF, increasing $\theta\ and\ \beta$ will cause the curve to rise more gradually, meaning it takes longer for the cumulative probability to reach 1.

The graphs of the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of the Weibull-Gamma distribution for different values of $\beta, \alpha\ and\ \theta$ are shown in Figures 1

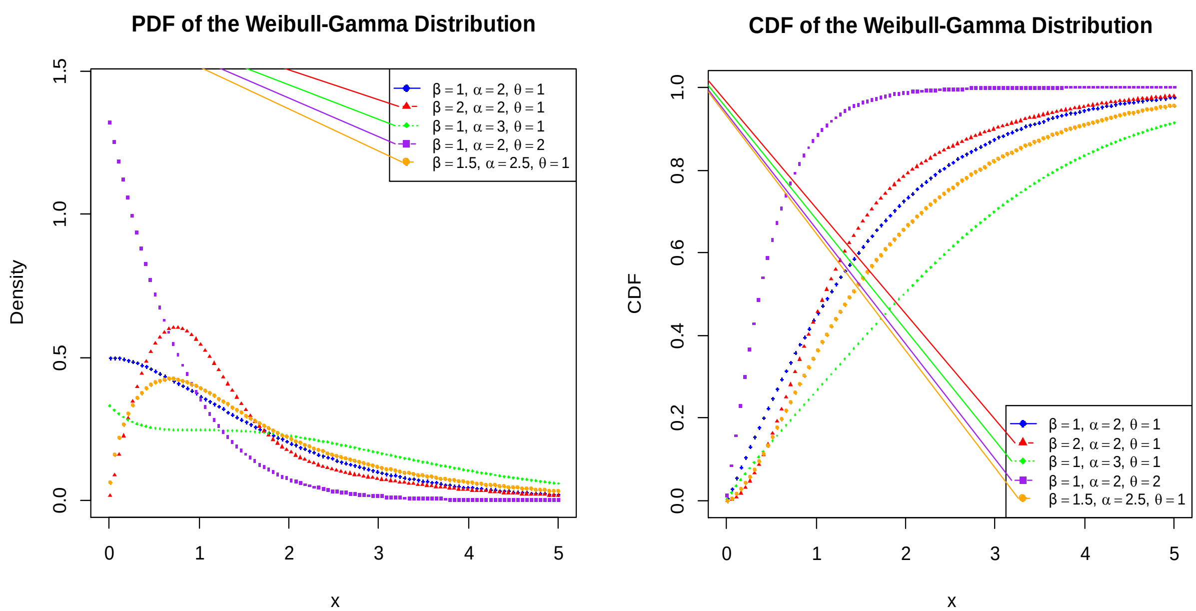

Figure 1: The graph of the p.d.f and c.d.f. of the WG distribution at different values of β, α and θ

Specific Effects on the Plot:

i. **For $\alpha = 1, \beta = 1$**,:
   a. This combination will gives the **Exponential distribution**. The PDF will have a simple exponential decay, and the CDF will rise steadily as x increases.

ii. **For $\alpha = 2, \beta = 1$**:

a. This combination will give the **Lindley distribution**. The PDF will be more spread out compared to the Exponential distribution, and the CDF will rise more slowly, indicating a more gradual accumulation of probability over x.

iii. **For $\boldsymbol{\beta} = \mathbf{1}$**:

a. With increasing in both α *and* θ causes the distribution to have a heavier tail. The PDF will decay slower, and the CDF will increase more gradually, showing that larger values of x are more probable.

iv. **For $\boldsymbol{\alpha} = \mathbf{1}$**,:

a. With both θ and β increasing, the PDF becomes more spread out, with a slower decay at the right tail. The CDF will rise more gradually, as the probability accumulates more slowly with respect to x.

From the density plot, it is clear that density plot of WG can take different shapes. For smaller α, the PDF peaks quickly and decays more rapidly while the CDF rises more sharply. As $\alpha$and β increase, the peak shifts to the right, and the tail becomes more spread out, with slower decay. Larger α, β, and θ values will cause the CDF to rise more gradually, indicating that values of x take longer to accumulate probability.It can be observed that the **PDF** for higher values of αor β spreads out, and the **CDF** for these same values rises more gradually.

Moment Generating Function

The moment generating function (MGF) of the new WG distribution can be derived from the standard definition of MGF as follows. The MGF of random variable x having the pdf $f(x;\beta,\alpha,\theta)$ is given by:

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tX} f(x;\beta,\alpha,\theta)dx \tag{16}$$

Now, with $f(x;\beta\ \alpha,\theta)$ being the WG density function in (5), then putting (5) in (16), the MGF of x is obtained as follows:

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tX}\frac{\theta\left[\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right) + (\theta x)^{\alpha-1}\exp(-\theta x)\right]}{\theta+\Gamma(\alpha)}dx$$

$$= \frac{\theta}{\theta+\Gamma(\alpha)}.\int_0^\infty e^{tX}\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right)dx + \frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}\int_0^\infty e^{tX}\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)}dx \tag{17}$$

Step 1: The first component, X_1, follows the Weibull distribution:

$$M_{X_1}(t) = \int_0^\infty e^{tX}\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right)dx = \sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{\Gamma\left(\frac{(k+\beta)}{\beta}\right)}{\theta^k} \tag{18}$$

Step 2: The second component, X_2 follows the Gamma distribution:

$$M_{X_2}(t) = \int_0^\infty e^{tX}\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)}dx = \left(1-\frac{t}{\theta}\right)^{-\alpha}, for\ t<\theta \tag{19}$$

Putting (18) and (19) in (17), we get:

$$M_X(t) = \frac{\theta}{\theta+\Gamma(\alpha)}.\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{\Gamma\left(\frac{(k+\beta)}{\beta}\right)}{\theta^k} + \frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}\left(1-\frac{t}{\theta}\right)^{-\alpha}$$

$$= \frac{\theta}{\theta+\Gamma(\alpha)}.\sum_{k=0}^{\infty}\frac{\left(\frac{t}{\theta}\right)^k}{k!}\left(\frac{k}{\beta}\right)! + \frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}\left(\frac{\theta}{\theta-t}\right)^{\alpha} \tag{20}$$

Laplace Transform of the new Weibull-Gamma (WG) Distribution

The Laplace transform of a probability density function (PDF), $f(x)$ is defined as:

$$\mathcal{L}_X(s) = \mathbb{E}[e^{-sX}] = \int_0^\infty e^{-sx}f(x)dx \tag{21}$$

Now, with $f(x;\beta\ \alpha,\theta)$ being the WG density function in (5), then putting (5) in (16), the MGF of x is obtained as follows:

$$\mathcal{L}_X(s) = \mathbb{E}(e^{-sX}) = \int_0^\infty e^{-sX}\frac{\theta\left[\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right) + (\theta x)^{\alpha-1}\exp(-\theta x)\right]}{\theta+\Gamma(\alpha)}dx$$

$$= \frac{\theta}{\theta+\Gamma(\alpha)}.\int_0^\infty e^{-sX}\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right)dx + \frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}\int_0^\infty e^{-sX}\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)}dx \tag{22}$$

Step 1: The first component, X_1, follows the Weibull distribution:

$$\mathcal{L}_{X_1}(s) = \int_0^\infty e^{-sX}\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right)dx = \sum_{k=0}^{\infty}\frac{(-s)^k}{k!}\frac{\Gamma\left(\frac{(k+\beta)}{\beta}\right)}{\theta^{\frac{k+\beta}{\beta}}} \quad (23)$$

Step 2: The second component, X_2 follows the Gamma distribution:

$$\mathcal{L}_{X_2}(s) = \int_0^\infty e^{-sX}\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)}dx = \left(1+\frac{s}{\theta}\right)^{-\alpha} = \left(\frac{\theta}{\theta+s}\right)^{\alpha} \quad (24)$$

Putting (23) and (24) in (22), we get:

$$\mathcal{L}_X(t) = \frac{\theta}{\theta+\Gamma(\alpha)}.\sum_{k=0}^{\infty}\frac{(-s)^k}{k!}\frac{\Gamma\left(\frac{(k+\beta)}{\beta}\right)}{\theta^{\frac{k+\beta}{\beta}}} + \frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}\left(\frac{\theta}{\theta+s}\right)^{\alpha} \quad (25)$$

Characteristic Function

The characteristic function (CF) of the new WG distribution can be derived from the standard definition of CF as follows. The CF of random variable x having the pdf $f(x;\beta,\alpha,\theta)$ is given by:

$$\Phi_X(t) = \mathbb{E}(e^{itX}) = \int_0^\infty e^{itX} f(x;\beta,\alpha,\theta)dx \quad (26)$$

Now, with $f(x;\beta\ \alpha,\theta)$ being the WG density function in (5), then putting (5) in (26), the CF of x is obtained as follows:

$$\Phi_X(t) = \mathbb{E}(e^{itX}) = \int_0^\infty e^{itX}\frac{\theta\left[\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right) + (\theta x)^{\alpha-1}\exp(-\theta x)\right]}{\theta+\Gamma(\alpha)}dx$$

$$= \frac{\theta}{\theta+\Gamma(\alpha)}.\int_0^\infty e^{itX}\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right)dx + \frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}\int_0^\infty e^{itX}\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)}dx \quad (27)$$

Step 1: The first component, X_1, follows the Weibull distribution:

$$\Phi_{X_1}(t) = \int_0^\infty e^{itX}\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right)dx = \sum_{k=0}^{\infty}\frac{(it)^k}{k!}\frac{\Gamma\left(\frac{(k+\beta)}{\beta}\right)}{\theta^{\frac{k+\beta}{\beta}}} \quad (28)$$

Step 2: The second component, X_2 follows the Gamma distribution:

$$\Phi_{X_2}(t) = \int_0^\infty e^{itX}\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)}dx = \left(\frac{\theta}{\theta-it}\right)^{\alpha} \quad (29)$$

Putting (28) and (29) in (27), we get:

$$\Phi_X(t) = \frac{\theta}{\theta+\Gamma(\alpha)}.\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\frac{\Gamma\left(\frac{(k+\beta)}{\beta}\right)}{\theta^{\frac{k+\beta}{\beta}}} + \frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}\left(\frac{\theta}{\theta-it}\right)^{\alpha} \quad (30)$$

The Moments and Related Measures

The rth moment about origin, μ_r', is defined as;

$$\mu_r' = \int_0^\infty x^r f(x;\beta,\alpha,\theta)dx \quad (31)$$

Then substituting (5) in (31), the rth moment about origin of the WG distribution is obtained as follows:

$$\mu_r' = \mathbb{E}(x^r) = \int_0^\infty x^r\frac{\theta\left[\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right) + (\theta x)^{\alpha-1}\exp(-\theta x)\right]}{\theta+\Gamma(\alpha)}dx$$

$$= \frac{\theta}{\theta+\Gamma(\alpha)}.\int_0^\infty x^r\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right)dx + \frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)}\int_0^\infty x^r\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)}dx \quad (32)$$

Step 1: The first component, X_1, follows the Weibull distribution:

$$E_1(X^r) = \int_0^\infty x^r\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right)dx = \frac{1}{\theta^r}\Gamma\left(\frac{r}{\beta}+1\right) \quad (33)$$

Step 2: The second component, X_2 follows the Gamma distribution:

$$E_2(X^r) = \int_0^\infty x^r\frac{\theta(\theta x)^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)}dx = \frac{\Gamma(\alpha+r)}{\theta^r\Gamma(\alpha)} \quad (34)$$

Putting (33) and (34) in (32), we get:

$$\mu_r' = \frac{\theta}{\theta+\Gamma(\alpha)} \cdot \frac{1}{\theta^r}\Gamma\left(\frac{r}{\beta}+1\right) + \frac{\Gamma(\alpha)}{\theta+\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+r)}{\theta^r\Gamma(\alpha)} \tag{35}$$

After simplification of (35)

$$\mu_r' = \mathbb{E}(X^r) = \frac{1}{\theta^r}\left[\frac{\theta\Gamma\left(\frac{r}{\beta}+1\right)}{\theta+\Gamma(\alpha)} + \frac{\Gamma(\alpha+r)}{\theta+\Gamma(\alpha)}\right] = \frac{\theta\Gamma\left(\frac{r}{\beta}+1\right)+\Gamma(\alpha+r)}{\theta^r(\theta+\Gamma(\alpha))}$$

Hence, the rth moment about origin is given as;

$$\mu_r' = \frac{\theta\Gamma\left(\frac{r}{\beta}+1\right)+\Gamma(\alpha+r)}{\theta^r(\theta+\Gamma(\alpha))};\ r = 1,2,3,4 \tag{36}$$

The first four moments about the origin are: substituting for $r = 1,2,3,4$ in (36) yields the first four moments about the origin of the WG distribution as:

$$\mu_1' = \frac{\theta\Gamma\left(\frac{1}{\beta}+1\right)+\Gamma(\alpha+1)}{\theta^1(\theta+\Gamma(\alpha))} = \frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}};\quad \mu_2' = \frac{\theta\Gamma\left(\frac{2}{\beta}+1\right)+\Gamma(\alpha+2)}{\theta^2(\theta+\Gamma(\alpha))} = \frac{\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!}{\theta^2\{\theta+(\alpha-1)!\}}$$

$$\mu_3' = \frac{\theta\Gamma\left(\frac{3}{\beta}+1\right)+\Gamma(\alpha+3)}{\theta^3(\theta+\Gamma(\alpha))} = \frac{\theta\left(\frac{3}{\beta}\right)!+(\alpha+2)!}{\theta^3\{\theta+(\alpha-1)!\}};\quad \mu_4' = \frac{\theta\Gamma\left(\frac{4}{\beta}+1\right)+\Gamma(\alpha+4)}{\theta^4(\theta+\Gamma(\alpha))} = \frac{\theta\left(\frac{4}{\beta}\right)!+(\alpha+3)!}{\theta^4\{\theta+(\alpha-1)!\}}$$

The moments about the mean: Using the relationship between moments, the moments about the mean of the distribution are obtained as; $\mu_r = E[(X-\mu)^r]$

The first moment about the mean is always zero: $\mu_1 = E[(X-\mu)^1] = E(X)-\mu = \mu-\mu = 0$

The second central moment (variance): $\mu_2 = E[X^2]-(E[X])^2 = \mu_2'-\mu^2$

$$\mu_2 = \frac{\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!}{\theta^2\{\theta+(\alpha-1)!\}} - \left\{\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right\}^2 = \frac{1}{\theta^2}\left[\frac{\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!}{\{\theta+(\alpha-1)!\}} - \left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\{\theta+(\alpha-1)!\}}\right)^2\right] \tag{37}$$

The third central moment (Skewness Numerator): $\mu_3 = \mu_3' - 3\mu\mu_2' + 2\mu^3$

$$\mu_3 = \frac{\theta\left(\frac{3}{\beta}\right)!+(\alpha+2)!}{\theta^3\{\theta+(\alpha-1)!\}} - 3\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)\left(\frac{\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!}{\theta^2\{\theta+(\alpha-1)!\}}\right) + 2\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)^3 \tag{38}$$

The 4th central moment (Kurtosis Numerator):$\mu_4 = \mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4$

$$\mu_4 = \left[\begin{array}{l} \frac{\theta\left(\frac{4}{\beta}\right)!+(\alpha+3)!}{\theta^4\{\theta+(\alpha-1)!\}} - 4\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)\left(\frac{\theta\left(\frac{3}{\beta}\right)!+(\alpha+2)!}{\theta^3\{\theta+(\alpha-1)!\}}\right) \\ +6\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)^2\left(\frac{\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!}{\theta^2\{\theta+(\alpha-1)!\}}\right) - 3\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)^4 \end{array}\right] \tag{39}$$

The coefficient of variation (CV), Skewness, $\sqrt{\beta_1}$, Kurtosis,β_2, and index of dispersion,γ, are thus presented as follows:

$$C.V = \frac{\sigma}{\mu} = \frac{\sqrt{\left((\theta+(\alpha-1)!)\left(\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!\right) - \left(\theta\left(\frac{1}{\beta}\right)!+\alpha!\right)^2\right)}}{\theta\left(\frac{1}{\beta}\right)!+\alpha!} \tag{39b}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{\frac{3}{2}}}=\frac{\frac{\theta\left(\frac{3}{\beta}\right)!+(\alpha+2)!}{\theta^3\{\theta+(\alpha-1)!\}}-3\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)\left(\frac{\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!}{\theta^2\{\theta+(\alpha-1)!\}}\right)+2\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)^3}{\left\{\frac{1}{\theta^2}\left[\frac{\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!}{\{\theta+(\alpha-1)!\}}-\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\{\theta+(\alpha-1)!\}}\right)^2\right]\right\}^{\frac{3}{2}}} \tag{40}$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{\left[\begin{array}{l}\frac{\theta\left(\frac{4}{\beta}\right)!+(\alpha+3)!}{\theta^4\{\theta+(\alpha-1)!\}}-4\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)\left(\frac{\theta\left(\frac{3}{\beta}\right)!+(\alpha+2)!}{\theta^3\{\theta+(\alpha-1)!\}}\right)\\+6\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)^2\left(\frac{\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!}{\theta^2\{\theta+(\alpha-1)!\}}\right)-3\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\theta\{\theta+(\alpha-1)!\}}\right)^4\end{array}\right]}{\left\{\frac{1}{\theta^2}\left[\frac{\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!}{\{\theta+(\alpha-1)!\}}-\left(\frac{\theta\left(\frac{1}{\beta}\right)!+\alpha!}{\{\theta+(\alpha-1)!\}}\right)^2\right]\right\}^2} \tag{41}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{\left(\theta\left(\frac{2}{\beta}\right)!+(\alpha+1)!\right)(\theta+(\alpha-1)!)-\left(\theta\left(\frac{1}{\beta}\right)!+\alpha!\right)^2}{\theta\left(\theta\left(\frac{1}{\beta}\right)!+\alpha!\right)(\theta+(\alpha-1)!)} \tag{42}$$

Survival Function: The **survival function $S(x)$**represents the probability that the random variable X takes a value greater than x. In other words, $S(x)$gives us the likelihood that X "survives" beyond a particular threshold x.

- **As $x \to 0$**: The survival function $S(x)$ is expected to be close to 1, as the probability that X exceeds 0 is very high (since Xstarts from 0).
- **As x increases**: The survival function will decrease because the probability that X exceeds x decreases as x increases. It will approach 0 as $x \to \infty$.

The Survival Function (SF), also known as the reliability Function is given by:

$$S(x)=1-F(x;\beta,\alpha,\theta), \qquad where\ F(x)\ \ is\ CDF \tag{43}$$

Putting (15) in (43), we get survival function of the new WG distribution as follows:

$$S(x)=1-\frac{\theta\left(1-\exp\left(-(\theta x)^\beta\right)\right)+\gamma(\alpha,\theta x)}{\theta+\Gamma(\alpha)}=\frac{\theta\exp\left(-(\theta x)^\beta\right)+\Gamma(\alpha)-\gamma(\alpha,\theta x)}{\theta+\Gamma(\alpha)} \tag{44}$$

Hazard Function

The Hazard Function (Failure Rate function) is given by:

$$h(x;\beta,\alpha,\beta)=\frac{f(x;\beta,\alpha,\theta)}{1-F(x;\beta,\alpha,\theta)}, where\ 1-F(x;\beta,\alpha,\theta)=S(x)\ is\ srvival\ fnction \tag{45}$$

Substituting for (5) and (44) in (45), we get hazard function of the new WG distribution as follows:

$$h(x;\beta,\alpha,\beta)=\frac{\theta\left[\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^\beta\right)+(\theta x)^{\alpha-1}\exp(-\theta x)\right]}{\theta\exp(-(\theta x)^\beta)+\Gamma(\alpha)-\gamma(\alpha,\theta x)} \tag{46}$$

The hazard function gives the instantaneous rate of failure (or hazard) at time x. It can provide insight into the shape and behavior of the distribution:

- If $h(x)$ is constant over time, this indicates a memoryless or exponential distribution.
- If $h(x)$ is increasing over time, it indicates that the failure rate increases as x increases, which is characteristic of distributions with "heavy" tails or distributions where the likelihood of an event increases over time.

If $h(x)$ is decreasing over time, it indicates that the failure rate decreases as x increases, which is common in systems where the risk of failure reduces as time progresses.

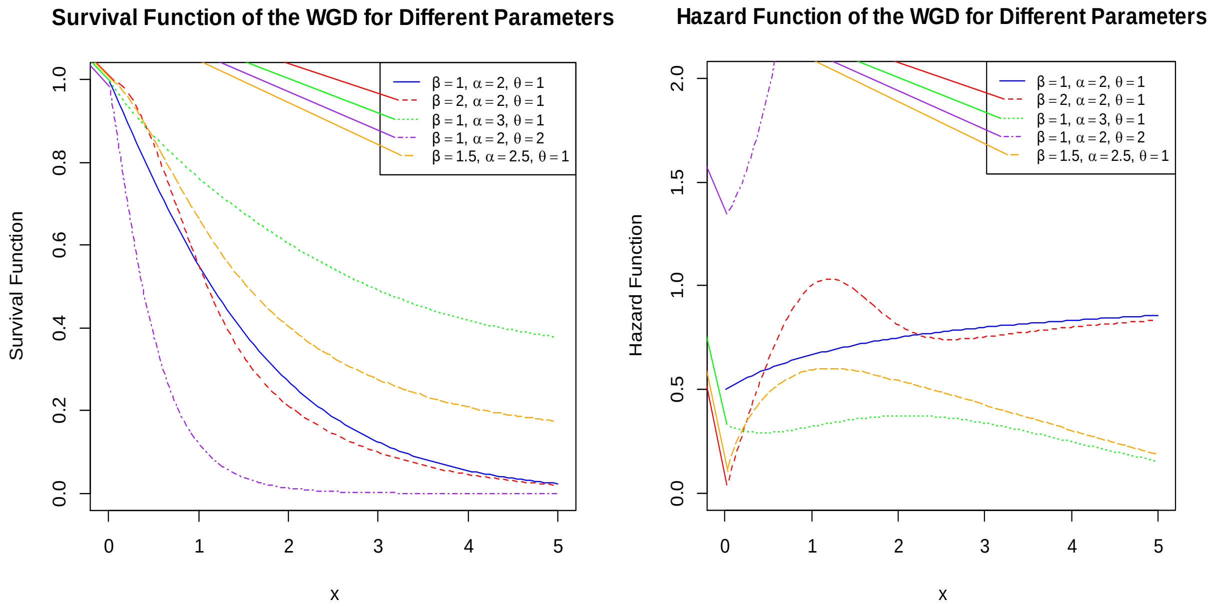

Figure 2: Graphs of the Survival and Hazard Functions of WGD at different values of $\boldsymbol{\beta}, \boldsymbol{\alpha}$ and $\boldsymbol{\theta}$

For small x, the survival function starts close to 1. This is because the probability that X exceeds a small value is high.As x increases, the survival function decreases, as the probability of X exceeding x becomes smaller. The rate of decrease depends on the model parameters α and β. Higher values of these parameters lead to slower decay in the survival function (indicating that the random variable X has a longer "tail"). Larger values of α, βmakes the survival function decay more slowly. This means that the "tail" of the distribution is heavier, and there is a higher chance that X takes larger values. This plot visually captures how the survival probability changes with different parameter values and show you the impact of the WG distribution's tail behavior.

Increasing Hazard Function: If the hazard function $h(x)$ increases as x increases, it indicates that the likelihood of "failure" increases as time goes on. This is often the case in distributions that model systems with aging or wear-out processes, where the risk increases over time.

Decreasing Hazard Function: If $h(x)$ decreases as x increases, it implies that the "failure rate" decreases over time. This may indicate systems that become more stable or resilient as time progresses.
By plotting the hazard function for different parameter values α, β, and θ, gives insights into how these parameters influence the failure rate over time

Order Statistics: Order statistics deals with properties and applications of ranked random variables. When it comes to studying natural problems related to flood, longevity, breaking strength, atmospheric pressure, wind etc., using order statistics becomes essential in the sense that the problem of interest in these cases reduces to that of extreme observations. Here, we provide the density of the kth order statistic$x_{k:n}$, say $g(y_{k:n;}\phi)$, in a random sample of size n from a new WG distribution. The expressions for rth raw moment, mean, and variance of the first and nth order statistics are also provided.

The probability density function of first-Order Statistics: The pdf of first-order statistic, denoted as $X_{(1)}$, represents the minimum value in a sample of size n drawn from a given probability density function (PDF). The distribution of $X_{(1)}$is characterized by its PDF and CDF, which we derive below.

Step 1: The CDF of $X_{(1)}$, denoted as $F_{X_{(1)}}(x)$ is given by:

$$F_{X_{(1)}}(x) = 1 - [\mathrm{S(x)}]^{\mathrm{n}}, \quad where\ S(x) = 1 - F(x) \tag{47}$$

Substituting (44) in (47), we have CDF of first-order of new WG distribution as follows:

$$F_{X_{(1)}}(x) = 1 - \left[\frac{\theta\exp\left(-(\theta x)^{\beta}\right) + \Gamma(\alpha) - \gamma(\alpha, \theta x)}{\theta + \Gamma(\alpha)}\right]^{\mathrm{n}} \tag{48}$$

Step 2: The PDF of $X_{(1)}$, is obtained by differentiating $F_{X_{(1)}}(x)$:

$$f_{X_{(1)}}(x) = \mathrm{n}[\mathrm{S(x)}]^{\mathrm{n}-1} f(x), \quad where\ f(x;\beta,\alpha,\theta)\ is\ new\ PDF\ of\ WG \tag{49}$$

Substituting (44) and (5) in (49), we have PDF of first-order of new WG distribution as follows:

$$f_{X_{(1)}}(x) = \mathrm{n}\left[\frac{\theta \exp\left(-(\theta x)^{\beta}\right) + \Gamma(\alpha) - \gamma(\alpha, \theta x)}{\theta + \Gamma(\alpha)}\right]^{\mathrm{n}-1} \left\{\frac{\theta\left[\beta\theta(\theta x)^{\beta-1} \exp\left(-(\theta x)^{\beta}\right) + (\theta x)^{\alpha-1} \exp(-\theta x)\right]}{\theta + \Gamma(\alpha)}\right\} \quad (50)$$

The probability density function of kth-Order Statistics: The pdf of kth order statistic, denoted as $X_{(k)}$, represents the kth smallest value in random sample of size n drawn from a given probability density function (PDF). The distribution of $X_{(k)}$is characterized by its PDF and CDF, which we derive below.

Step 1: The cumulative distribution function (CDF) of $X_{(k)}$, denoted as $F_{X_{(k)}}(x)$ is given by:

$$F_{X_{(k)}}(x) = \mathrm{P}\left(\mathrm{X}_{(k)} \leq \mathrm{x}\right) = \sum_{j=k}^{n} \binom{n}{j} [F(x)]^{j} [S(x)]^{n-j}; \;\; where\ S(x) = 1 - F(x) \& F(x) is\ CDF \quad (51)$$

Substituting (44) and (15) in (51), we have CDF of kth order of new WG distribution as follows:

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} \left[\frac{\theta\left(1 - \exp\left(-(\theta x)^{\beta}\right)\right) + \gamma(\alpha, \theta x)}{\theta + \Gamma(\alpha)}\right]^{j} \left[\frac{\theta \exp\left(-(\theta x)^{\beta}\right) + \Gamma(\alpha) - \gamma(\alpha, \theta x)}{\theta + \Gamma(\alpha)}\right]^{n-j} \quad (52)$$

Step 2: The PDF of $X_{(k)}$, is obtained by differentiating $F_{X_{(k)}}(x)$:

$$f_{X_{(k)}}(x) = \mathrm{n} \binom{n-1}{k-1} [\mathrm{F(x)}]^{\mathrm{k}-1} [\mathrm{S(x)}]^{\mathrm{n}-\mathrm{k}} f(x), \quad (53)$$

Substituting (44), (5) and (15) in (53), we have PDF of kth order of new WG distribution as follows:

$$f_{X_{(k)}}(x) = \left\{\begin{array}{c} \mathrm{n}\binom{n-1}{k-1}\left[\frac{\theta\left(1 - \exp\left(-(\theta x)^{\beta}\right)\right) + \gamma(\alpha, \theta x)}{\theta + \Gamma(\alpha)}\right]^{\mathrm{k}-1} \left[\frac{\theta \exp\left(-(\theta x)^{\beta}\right) + \Gamma(\alpha) - \gamma(\alpha, \theta x)}{\theta + \Gamma(\alpha)}\right]^{n-k} \\ \times\left(\frac{\theta\left[\beta\theta(\theta x)^{\beta-1} \exp\left(-(\theta x)^{\beta}\right) + (\theta x)^{\alpha-1} \exp(-\theta x)\right]}{\theta + \Gamma(\alpha)}\right) \end{array}\right\}$$

$$= \left\{\frac{\mathrm{n}\binom{n-1}{k-1}}{\left(\theta + \Gamma(\alpha)\right)^{n}} \left(\begin{array}{c} \left(\theta\left(1 - \exp\left(-(\theta x)^{\beta}\right)\right) + \gamma(\alpha, \theta x)\right)^{k-1} \left(\theta \exp\left(-(\theta x)^{\beta}\right) + \Gamma(\alpha) - \gamma(\alpha, \theta x)\right)^{n-k} \\ \times\left(\theta\left[\beta\theta(\theta x)^{\beta-1} \exp\left(-(\theta x)^{\beta}\right) + (\theta x)^{\alpha-1} \exp(-\theta x)\right]\right) \end{array}\right)\right\} \quad (54)$$

The probability density function of nth Order Statistics: The pdf of nth order statistic, denoted as $X_{(n)}$, represents the minimum value in a sample of size n drawn from a given probability density function (PDF). The distribution of $X_{(n)}$is characterized by its PDF and CDF, which we derive below.

Step 1: The CDF of $X_{(n)}$, denoted as $F_{X_{(n)}}(x)$ is given by:

$$F_{X_{(n)}}(x) = \mathrm{P}\left(X_{(n)} \leq \mathrm{x}\right) = [\mathrm{F(x)}]^{\mathrm{n}}, \qquad where\ F(x)\ is\ the\ CDF\ of\ new\ WG\ distribution \quad (55)$$

Substituting (15) in (55), we have CDF of first-order of new WG distribution as follows:

$$F_{X_{(n)}}(x) = \left[\frac{\theta\left(1 - \exp\left(-(\theta x)^{\beta}\right)\right) + \gamma(\alpha, \theta x)}{\theta + \Gamma(\alpha)}\right]^{\mathrm{n}} \quad (56)$$

Step 2: The PDF of $X_{(n)}$, is obtained by differentiating $F_{X_{(n)}}(x)$:

$$f_{X_{(n)}}(x) = \mathrm{n}[\mathrm{F(x)}]^{\mathrm{n}-1} f(x), \qquad where\ f(x; \beta, \alpha, \theta)\ is\ new\ PDF\ of\ WG \quad (57)$$

Substituting (15) and (5) in (57), we have PDF of kth order of new WG distribution as follows:

$$f_{X_{(n)}}(x) = \mathrm{n}\left[\left[\frac{\theta\left(1 - \exp\left(-(\theta x)^{\beta}\right)\right) + \gamma(\alpha, \theta x)}{\theta + \Gamma(\alpha)}\right]\right]^{\mathrm{n}-1} \left\{\frac{\theta\left[\beta\theta(\theta x)^{\beta-1} \exp\left(-(\theta x)^{\beta}\right) + (\theta x)^{\alpha-1} \exp(-\theta x)\right]}{\theta + \Gamma(\alpha)}\right\}$$

$$= n \frac{\left(\theta\left(1 - \exp\left(-(\theta x)^{\beta}\right)\right) + \gamma(\alpha, \theta x)\right)^{n-1} \left(\theta\left[\beta\theta(\theta x)^{\beta-1} \exp\left(-(\theta x)^{\beta}\right) + (\theta x)^{\alpha-1} \exp(-\theta x)\right]\right)}{\left(\theta + \Gamma(\alpha)\right)^{n}} \quad (58)$$

Parameters Estimation

Maximum Likelihood Estimate

Let x_i , $i = 1, 2, 3, \ldots n$, be a random sample of size n from the Weibull-Gamma distribution.

Step 1: Define the likelihood function:

Given a random sample X_1, $X_2, \ldots, X_n$,from the new Weibull-Gamma (WG) distribution, the likelihood function, L is defined by;

$$L(\beta,\alpha,\theta)=\prod_{i=1}^{n} f(X_i|\beta,\alpha,\theta) \quad (59)$$

Putting (5) in (59), the likelihood function of the WG distribution is obtained as;

$$L(\beta,\alpha,\theta)=\prod_{i=1}^{n}\left(\frac{\theta\left[\beta\theta(\theta x)^{\beta-1}\exp\left(-(\theta x)^{\beta}\right)+(\theta x)^{\alpha-1}\exp(-\theta x)\right]}{\theta+\Gamma(\alpha)}\right)$$

$$=\left(\frac{\theta}{\theta+\Gamma(\alpha)}\right)^{n}(\beta\theta)^{n}\sum_{i=1}^{n}(\theta x_i)^{\beta}\exp\left[-\sum_{i=1}^{n}(\theta x_i)^{\beta}\right]+\sum_{i=1}^{n}(\theta x_i)^{\alpha-1}+\exp\left[-\sum_{i=1}^{n}(\theta x_i)^{\beta}\right] \quad (60)$$

Step 2: Construct the log-likelihood function: we take the natural log-of likelihood function in (60)

$$\ell(\beta,\alpha,\theta)=\log\left\{\left(\frac{\theta}{\theta+\Gamma(\alpha)}\right)^{n}(\beta\theta)^{n}\sum_{i=1}^{n}(\theta x_i)^{\beta}\exp\left[-\sum_{i=1}^{n}(\theta x_i)^{\beta}\right]+\sum_{i=1}^{n}(\theta x_i)^{\alpha-1}+\exp\left[-\sum_{i=1}^{n}(\theta x_i)^{\beta}\right]\right\}$$

$$=2n\log\theta-n\log\left(\theta+\Gamma(\alpha)\right)+n\log\beta-\sum_{i=1}^{n}(\theta x_i)^{\beta}+(\alpha-1)\log\sum_{i=1}^{n}(\theta x_i)-\sum_{i=1}^{n}(\theta x_i) \quad (61)$$

Differentiating (61) partially with respect to β, we have;

$$\frac{\partial\ell(\beta,\alpha,\theta)}{\partial\beta}=0\Rightarrow\frac{n}{\beta}-\sum_{i=1}^{n}(\theta x_i)^{\beta}\log\sum_{i=1}^{n}(\theta x_i)=0\Rightarrow\beta\sum_{i=1}^{n}(\theta x_i)^{\beta}=\frac{n}{\log\sum_{i=1}^{n}(\theta x_i)} \quad (62)$$

Furthermore, differentiating (61) partially with respect to α, we have;

$$\frac{\partial\ell(\beta,\alpha,\theta)}{\partial\alpha}=0\Rightarrow-\frac{n}{\theta+\Gamma(\alpha)}.\Gamma'(\alpha)+\log\sum_{i=1}^{n}(\theta x_i)=0$$

$$\Rightarrow\psi(x)=\left(\frac{\theta+\Gamma(\alpha)}{n\Gamma(\alpha)}\right)\sum_{i=1}^{n}(\theta x_i)\ ; \quad where\ \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}=\psi(x)\ is\ the\ digamma\ function\ of\ \alpha \quad (63)$$

Finally, differentiating (61) partially with respect to θ, we have;

$$\frac{\partial\ell(\beta,\alpha,\theta)}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{\theta+\Gamma(\alpha)}-\frac{\beta}{\theta}.\sum_{i=1}^{n}(\theta x_i)^{\beta}+\frac{\alpha-1}{\theta}-\sum_{i=1}^{n}x_i=0\ [multiply\ all\ through\ by\ \theta(\theta+\Gamma(\alpha))]$$

$$\Rightarrow 2n\left(\theta+\Gamma(\alpha)\right)-n\theta-\beta\left(\theta+\Gamma(\alpha)\right)\sum_{i=1}^{n}(\theta x_i)^{\beta}+(\alpha-1)\left(\theta+\Gamma(\alpha)\right)-\theta\left(\theta+\Gamma(\alpha)\right)\sum_{i=1}^{n}x_i=0 \quad (64)$$

The partial derivatives of the likelihood function in (34) with respect to parameters α and θ cannot be solved analytically because they are not in closed forms. These equations can be solved using any numerical method such as the Newton-Raphson method (Henningsen & Toomet, 2011), the Nelder-Mead method (Nelder & Mead, 1965), BFGS method (Fletcher, 1987), SANN method (Belisle, 1992), and the like. The Newton-Raphson method is however employed in this paper with the use of the "optim" function in *R* package (R Core Team, 2018) and Henningsen & Toomet, (2011) to solve the equations iteratively.

Applications

In this section, the goodness-of-fit of the new WG distribution is discussed with an application to real-life datasets. The parameters of the distribution were solved using the MLE method while the goodness-of-fit was evaluated using the Akaike Information Criterion (AIC), Akaike Information Criterion Corrected (AICC), Bayesian Information Criterion (BIC) and –2logLik with their respective statistics given below.

$$AIC=-2\ln L+2k \quad (65)$$

$$AICC=AIC+\frac{2k(k+1)}{n-k-1} \quad (66)$$

$$BIC=-2\ln L+k\ln n \quad (67)$$

where k is the number of parameters and n is the sample size. The distribution that has a lower value of these criteria is judged to be the best among others.

Data Description

Dataset 1: This dataset consists of the Exceedances of flood peaks (in m^3/s) of the Wheaton Rivernear Carcross in Yukon Territory, Canada for the years 1958-1984 reported in Akinsete & Famoye (2008).

Dataset 2: This represents an uncensored dataset corresponding to Remission times (in months) of a random sample of 118 Bladder cancer patients reported in Lee and Wang (2003).

RESULTS AND DISCUSSION

Table 1: Parameters Estimation and Goodness-of-Fit Test Results of the WGD and other Existing distributions for the Exceedances of Wheat on River flood data

	EXP.	GAMMA	WD	LINDLEY	E-Gamma	WGD
ML Estimates	$\hat{\theta}$=0.0819	$\hat{\alpha}$=0.8383	$\hat{\theta}$=0.9296	$\hat{\theta}$=0.0619	$\hat{\alpha}$=1.4616	$\hat{\beta}$=0.0938
		$\hat{\theta}$=0.0687	$\hat{\beta}$=0.1108		$\hat{\theta}$=0.0755	$\hat{\alpha}$=0.0160
						$\hat{\theta}$=3.1105
-2logLik	504.256	502.689	481.902	430.051	499.012	481.425
AIC	506.256	506.689	485.902	432.051	503.012	487.425
AICC	506.313	506.863	486.076	432.108	503.186	487.749
BIC	508.533	506.965	486.178	434.327	503.289	490.255

Table 2: Parameters Estimation and Goodness-of-Fit Test Results of the WGD and other Existing distributions for the Remission times (in months) of Bladder cancer data

	EXP.	GAMMA	WD	LINDLEY	EXP-Gamma	WGD
ML Estimates	$\hat{\theta}$=0.1163	$\hat{\alpha}$=0.3856	$\hat{\theta}$=1.0478	$\hat{\theta}$=0.1961	$\hat{\alpha}$=1.4616	$\hat{\beta}$=0.0852
		$\hat{\theta}$=0.7437	$\hat{\beta}$=0.1046		$\hat{\theta}$=0.1041	$\hat{\alpha}$=1.0473
						$\hat{\theta}$=2.078
-2logLik	743.718	743.159	728.228	839.060	742.679	721.084
AIC	745.718	747.159	734.228	841.060	746.679	727.084
AICC	745.753	747.263	734.422	841.095	746.783	727.187
BIC	748.489	747.930	742.784	843.912	747.449	734.209

An analysis of the remission times for the Exceedances of Wheaton River flood and bladder cancer datasets using various lifetime distributions revealed that the Weibull-Gamma (WG) distribution provided the best fit among all models considered. Tables 1 and 2 present the maximum likelihood (ML) estimates of the parameters for each distribution, along with their corresponding values of -2 log-likelihood (-2logLik), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC), for comparative evaluation.

For the Exceedances of Wheaton River flood data, the WG distribution—with its three parameters—achieved the lowest values of -2logLik (481.425), AIC (487.425), and BIC (490.255), indicating a superior model fit. Similarly, for the bladder cancer remission time data, the WG distribution again recorded the lowest values of -2logLik (721.084), AIC (727.084), and BIC (734.209), confirming its robustness across different datasets. These low AIC and BIC values indicate not only a good fit but also model parsimony, which is critical in balancing complexity with predictive accuracy.

Other distributions examined—including the Exponential Distribution (ED), Gamma Distribution (GD), Weibull Distribution (WD), Lindley Distribution (LD), and Exponential-Gamma Distribution (EGD)—showed comparatively higher AIC and BIC values, suggesting they were less suitable for modeling the remission times in both datasets.

Regarding parameter estimates, the WG distribution's values suggest a relatively slow remission rate, which aligns with typical medical expectations, where remission times often show a gradual decline. The distribution's shape and scale parameters provide valuable insights into the underlying data structure, allowing for a more detailed understanding of remission dynamics.

The superior performance of the WG distribution highlights its robustness and flexibility in modeling lifetime data, particularly for complex datasets such as cancer remission times. Its ability to accommodate varying hazard rate behaviors makes it well-suited for survival analysis where the hazard function is non-constant.

Moreover, the favorable AIC and BIC results underscore the importance of considering both model fit and parsimony in statistical modeling. The WG distribution's effective balance between these factors makes it a practical and powerful tool for researchers and practitioners in fields such as oncology, where accurate modeling of survival and remission times is crucial for informed decision-making.

While the WG distribution demonstrated strong performance in this study, there remain opportunities for further research and development.

CONCLUSION

This paper introduced and examined the Weibull-Gamma (WG) distribution, a highly flexible model capable of capturing a wide range of data behaviors, including heavy tails, skewness, and varying degrees of variability, depending on its parameters. Due to its versatility, the WG distribution is well-suited for applications in reliability analysis, survival analysis, queuing theory, and financial modeling.

The findings of this study highlight the WG distribution as a powerful and adaptable tool for modeling lifetime data. Its effectiveness was particularly evident in real-world datasets such as cancer remission times and the Exceedances of Wheaton River flood data. The WG distribution consistently achieved superior model fit, as indicated by its lower AIC and BIC values compared to traditional distributions. These results suggest that the WG distribution is a valuable addition to the statistical modeling toolkit, especially in survival and reliability contexts where conventional models like the Weibull or Gamma distributions may fall short in capturing the data's underlying complexity.

Given its promising performance, the WG distribution emerges as a strong candidate for modeling scenarios involving non-standard hazard rates or heterogeneous data structures.

Future studies could investigate the sensitivity of the model to initial parameter estimates and explore its performance across datasets of varying sizes. Additionally, it would be beneficial to examine the model's applicability to other medical data types—such as mortality rates or recovery times—to validate its generalizability.

Another promising direction for future research involves extending the WG model by incorporating covariates such as age, treatment regimen, or genetic markers. This would enhance the model's explanatory power and support the development of personalized treatment strategies. Furthermore, the creation of efficient computational tools and algorithms for parameter estimation in large-scale clinical trials would facilitate broader application of the WG distribution, enabling more accurate modeling across diverse medical and clinical datasets. Future research should further explore the practical applications and theoretical extensions of the WG distribution, particularly in clinical and biomedical contexts where individualized treatment decisions are becoming increasingly important. Additionally, developing a generalized version of the Weibull-Gamma distribution could enhance its modeling capabilities and broaden its applicability to even more complex datasets.

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