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On Singh's Dressed Epsilon Perspective of Multigroup

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ABSTRACT
This paper investigates some aspects of multigroups from Singh's perspective. It introduces a structured approach to analyzing the aspects of multigroups presented in this paper using dressed epsilon notation. We begin by defining the hierarchical decomposition of multisets, establishing that each r-level reference set in the hierarchical decomposition of a multiset over a group is itself a subgroup.We then present a fundamental characterization of multigroups by proving that a multiset$A$ is a multigroup over a set X if and onlyif $xy^{-1} \in^p A, \forall x \in^m A \Longrightarrow p \geq (m \wedge n)$. Additionally, we define the sets A^* and $A_*$and prove that both are subgroups of X using Singh's dressed epsilon notation. Our work further investigates the algebraic properties of multigroupsand establishes criteria for commutativity. We also demonstrate that while the intersection of two multigroups is always a multigroup, their union does not necessarily inherit this structure. The concept of submultigroup is introduced to formalize the relationship between two multigroups. Finally, we establish the equivalence between certain multigroup properties, such as the symmetry of multisets based on product of elements and conjugate conditions.

Keywords:
Dressed epsilon,
Multigroup Operations,
Singh,
Submultigroup.

INTRODUCTION

The concept of groups forms a fundamental pillar in abstract algebra, with applications spanning fields such as mathematics, computer science and physics. Classical group theory primarily deals with sets and binary operations that satisfy closure, associativity, the existence of an identity element, and the presence of an inverse. However, there is a growing interest in generalizing these notions to accommodate multisets, leading to the concept of multigroups (Baumslag and Chandler, 1968).

Singh introduced the notion of a multigroup as a natural extension of the classical group, utilizing the dressed epsilon notation $(\in_+, \in^{\boldsymbol{n}}, \in_+^{\boldsymbol{n}})$ to capture the multiplicity of elements within the group structure. The dressed epsilon notation provides a precise way to express that an element belongs to a multiset with a certain multiplicity, thereby facilitating the extension of group operations to multisets. The motivation behind studying multigroups lies in their potential to model complex systems where redundancy and repetition are inherent, such as in computational and combinatorial contexts. (Singh, 2006 and Singh et al., 2008).

In this paper, we explore the fundamental properties and structure of multigroups within the framework of group theory. We begin by establishing the hierarchical decomposition of multisets, providing a systematic way to organize elements based on their multiplicity levels. This decomposition is crucial for understanding how multigroups can be decomposed into structured subgroups (Singh and Isah, 2016).

We then delve into the properties of multigroups, presenting necessary and sufficient conditions for a multiset to qualify as a multigroup. We investigate the product and inverse conditions that characterize multigroups and examine how these properties extend from traditional group axioms. Furthermore, we explore the algebraic relationships between multigroups, including the commutative properties of multigroup multiplication, and provide conditions under which the product of two multigroups remains a multigroup. We also examine intersection and union operations on multigroups, highlighting scenarios where these operations preserve the multigroup structure.(Ejegwa and Ibrahim, 2020).

Through this study, we aim to enrich the theoretical foundation of multigroups by identifying their structural characteristics, algebraic properties, and potential applications. Our results not only bridge the gap between classical group theory and multiset theory but also open avenues for applying multigroup concepts to real-world problems where element repetition plays a significant role (Peter, et al., 2025).

Recent studies have further enriched the understanding of multigroups. Nazmul et al. (2013) introduced foundational concepts related to multigroups derived from multisets. Awolola and Ibrahim (2016) explored various properties of multigroups, while Awolola and Ejegwa (2017) examined the order of elements within these structures. Awolola (2019) investigated cyclic multigroup families, shedding light on their structural characteristics. Additionally, Ibrahim et al. (2016) provided a comprehensive overview of multigroup theory and its potential applications.

MATERIALS AND METHODS

i. This study employs a theoretical approach to investigating the properties and structures of multigroups within the framework of multiset theory.
ii. The research methodology encompasses the following key steps. Previous works on multigroups are analyzed.
iii. The study established a theoretical framework for multigroups. This included defining multigroups in the context of multisets and formulating precise definitions and notations to capture the multiplicity of elements within these structures.
iv. A series of propositions were formulated to characterize the algebraic properties of multigroups. Each proposition was accompanied by rigorous mathematical proofs to validate the conditions under which a multiset qualifies as a multigroup.
v. The study investigated various operations within multigroups, including multiplication, inversion, intersection, and union. The closure properties of these operations were examined to determine how they affect the multigroup structure.
vi. Specialized substructures within multigroups, were defined and analyzed. The study explored the conditions under which these substructures form subgroups and their significance in the broader context of multigroup theory.

Preliminary Definitions

Definition 1 (Multiset) A multiset M over a domain set$E$ is a collection of elements of E with repetitions allowed. The set E is called the ground or generic set of the class of all multisets containing elements from E.

Definition 2 (Union) Suppose M and $N$are two multisets over a ground set S, then $A \cup B$ is the multiset defined by $x \in^k (M \cup N) \Leftrightarrow x \in^m M$ and $x \in^n N \Longrightarrow\ k = \max(m, n)$.

Definition 3 (Intersection) Suppose$M$ and $N$are two multisets over a ground set S, then $A \cup B$ is the multiset defined by $x \in^k (M \cup N) \Leftrightarrow x \in^m M$ and $x \in^n N \Longrightarrow k = \min(m, n)$.

Definition 4 Let A and B be multigroups over a group X. The product $A \circ B$ is defined such that for any element $x \in^+ A \circ B$, there exist elements $a \in^+ A$ and $b \in^+ B$ such that $x = a \cdot b$ where $\cdot$ denotes the group operation.

RESULTS AND DISCUSSION

Definition 5 (Multigroup) Let Xbe a group. A multiset$G$ over X is said to be a multigroup over X if it satisfies the following two conditions:

(i) $x \in^m G \wedge y \in^n G \Longrightarrow xy \in_+^{(m \wedge n)} G\ \forall x, y \in X$; (Multiplication condition)

(ii) $x \in^n G \Longrightarrow x^{-1} \in_+^n G\ \forall x \in X$; (Inverse condition)

Example 1

For example, consider the cyclic group of order 4 $X = \{e, a, a^2, a^3\}$ be the cyclic group of order 4, where $a^4 = e$. Let the multiset $G = \{e, e, e, a, a, a^2, a^2, a^3, a^3\}$ be a multiset over X. The membership conditions are given as follows:

Multiplication conditions:

For all $x, y \in X$, we verify that the multiplication condition

$$x \in^m G \wedge y \in^n G \Longrightarrow xy \in_+^{(m \wedge n)} G\ \forall x, y \in X \quad (1)$$

holds:

$ea \in^2 G$, since $a \in^3 G$ and $e \in^3 G$, so

$ea = a \in_+^{(3 \wedge 3)} G$.

$ea^2 \in^2 G$, since $a^2 \in^2 G$ and $e \in^3 G$, so

$ea^2 = a^2 \in_+^{(3 \wedge 2)} G$.

$ea^3 \in^2 G$, since$a^3 \in^2 G$ and $e \in^3 G$, so

$ea^3 = a^3 \in_+^{(3 \wedge 2)} G$.

$aa^2 \in^2 G$, since $a \in^3 G$ and$a^2 \in^2 G$, so

$aa^2 = a^3 \in_+^{(3 \wedge 2)} G$.

$aa^3 \in^2 G$, since $a \in^3 G$ and$a^3 \in^2 G$, so

$aa^3 = e \in_+^{(3 \wedge 2)} G$.

$a^2a^2 \in^2 G$, since$a^2 \in^2 G$ so

$a^2a^2 = e \in_+^{(2 \wedge 2)} G$.

$a^2a^3 \in^2 G$, since$a^2 \in^2 G$ and$a^3 \in^2 G$, so

$a^2a^3 = a \in_+^{(2 \wedge 2)} G$.

$a^3a^3 \in^2 G$, since$a^3 \in^2 G$, so

$a^3a^3 = a^2 \in_+^{(2 \wedge 2)} G$.

Inversion condition

For all $x \in X$, we verify:

$$x \in^n G \Longrightarrow x^{-1} \in_+^n G \quad (2)$$

$a^{-1} = a^3$, and $a^3 \in^2 G \Longrightarrow a^{-1} \in_+^2 G$

$(a^2)^{-1} = a^2 \in_+^2 G$

$(a^3)^{-1} = a$, and $a \in^2 G \Longrightarrow (a^3)^{-1} \in_+^2 G$

$e^{-1} = e$, and $e \in^3 G \Longrightarrow e^{-1} \in_+^3 G$

Since both conditions hold, we conclude that G is a multigroup over X.

Proposition 1

Let A be a multiset. Then A is a multigroup over a set X if and only if $xy^{-1} \in^p A \; \forall x \in^n A$ and $y \in^m A \Longrightarrow p \geq \min(m, n)$

Proof

Assume A is a multigroup over X, By definition of a multigroup, we have the following two conditions:

1. Multiplication condition: If $x \in^m A$ and $y \in^n A$ then $xy \in_+^{(m \wedge n)} A$.
2. Inverse condition: If $x \in^n A$, then $x^{-1} \in_+^n A$.

Since A is a multigroup, for any $x, y \in X$, we have: $y \in^m A \Longrightarrow y^{-1} \in_+^n A$ (by the inverse) condition. $x \in^n A$ and $y^{-1} \in_+^n A \Longrightarrow xy^{-1} \in_+^{(m \wedge n)} A$ (by the multiplication condition). Thus, $xy^{-1} \in_+^{(m \wedge n)} A$ as required.

Conversely, assume that for any $x, y \in X$ such that $x \in^n A$ and $y \in^m A$, we have $xy^{-1} \in_+^{(m \wedge n)}$. We need to show that A satisfies the multiplication and inverse conditions of a multigroup. Now take $y = x$. Then, $x \in^n A \Longrightarrow xx^{-1} = e \in_+^n A$, where e is the identity element . Hence, $x^{-1} \in_+^n A$. Thus, the inverse condition is satisfied.

Now, for any $x, y \in X$, we know from the hypothesis, that $xy^{-1} \in_+^{(m \wedge n)} A$. Sincethe group operation is closeand the inverse condition holds, then multiplication condition also holds.

Definition 6 (Hierarchical decomposition of multisets) Let M be a mutiset over a set X, then the set $M_r = \{x \in X : x \in_+{}^r M\}$ is called r-level reference of M where r is the position of the reference set when all the reference sets (the empty set inclusive) are arranged in a descending order using the non-proper containment relation $\supseteq$. In this case, the set M_r for each r is known as an r*-reference set*.

Proposition 2

Let M be a multiset over a group X. The r-level reference sets M_r in the hierarchical decomposition of multiset M are subgroups of X.

Proof

Since X is a group, we need to show that for any $x, y \in M_r$, the element xy^{-1} also belongs to M_r. Since $x, y \in M_r$, it means that $x \in_+^r M$ and $y \in_+^r M$. By the definition of a multigroup

$$x \in^m A \wedge y \in^n A \Longrightarrow xy \in_+^{(m \wedge n)} A \quad (3)$$

Since both x and y appear at least r times in M, we have that $m, n \geq r$. Therefore, $m \wedge n \geq r$. Thus,

$$xy^{-1} \in_+^{(m \wedge n)} M \Longrightarrow xy^{-1} \in_+^r M \quad (4)$$

. This shows that $xy^{-1} \in M_r$.

Defnition 7 Let A be a multigroup over a group X. Define A^* and A_* as

$$A^* = \{x \in X : x \in^m A \wedge e \in^n A \Longrightarrow m = n\}$$

and

$$A_* = \{x \in X : x \in^m A \Longrightarrow m > 0\}$$

Proposition 3 Let A be a multigroups over a group X then A^* and A_* are submultigroups of A.

Proof

A is a subgroup, the identity element e of the group X is also in A with some multiplicity n. By the definition of A^*, for any $x \in A^*$, $x \in^m A$ and $e \in^n A$ imply $m = n$. Hence, the identity element e satisfies the condition and is in A^*.

Take any $x, y \in A^*$. Then $x \in^m A$ and $y \in^n A$ imply $m = n$. Since A is a multigroup, the product xy^{-1} must also be in A with multiplicity at least $(m \wedge n)$, which in this case is equal to m since $m = n$. Thus, $xy^{-1} \in A^*$. Therefore, A^* is a subgroup of A.

Since A is a multigroup and e is the identity, e must be present in A with a positive multiplicity. Hence, $e \in A^*$.

Take any $x, y \in A_*$. Then $x \in^m A$ and $y \in^n A$ imply $m \geq$ and $n > 0$. Since A is a multigroup, the product xy^{-1} must be in A with multiplicity at least $(m \wedge n)$, which is strictly positive. Therefore, $xy^{-1} \in A_*$. Therefore, A_* is a subgroup of A.

Proposition 4 A multiset A is a multigroup over a group X if and only if the following properties are satisfied:

i. $A \circ A \subseteq A$;

ii. $A^{-1} \subseteq A$

Proof
Assume that A is a multigroup over a group X. Now for any $x, y \in X$ such that $x \in^m A$ and $y \in^n$, $xy \in_+^{(m \wedge n)} A$. This implies that the product of any two elements from A is still in A with at least the minimum multiplicity. Therefore, $A \circ A \subseteq A$.
Also, for any $x \in X$ such that $x \in^m A$, $x^{-1} \in_+^m$. This means that the inverse of any element from A is also in A with the same multiplicity, hence, $A^{-1} \subseteq A$
Conversely, since the first condition states that the product of any two elements from A is still in A, it satisfies the multiplicity condition of a multigroup. The second condition states that the inverse of any element in A is also in A, satisfying the inverse ondition of a multigroup.

Proposition 5 A multiset$A$ is a multigroup over a group X if and only if the following properties are satisfied $A \circ A^{-1} \subseteq A$.
Proof
Assume that A is a multigroup over a group X. By the definition of a multigroup, For any $x, y \in X$ such that $x \in^m A$ and $y \in^n$, $xy \in_+^{(m \wedge n)} A$. This implies that the product of any element from A and the inverse of any other element from A is still in A with at least the minimum multiplicity. Therefore, $A \circ A^{-1} \subseteq A$.
Conversely, Assume that the given property holds: that is, $A \circ A^{-1} \subseteq A$. Since this condition states that the product of any element form A is still in A, it satisfies the multiplication condition of a multigroup. Furthermore, the inverse condition of a multigroup states that for any $x \in^m A$, the inverse $x^{-1} \in_+^m$, which follows from the fact that the set is closed under taking inverses.

Proposition 6 Let A and B be multigroups over a group X. Then $A \circ B$ is a multigroup over X if and only if $A \circ B = B \circ A$.
Proof
Assume $A \circ B$ is a multigroup, it must satisfy the multiplication condition and the invese condition. By the multiplication condition for multigroups: For any $x, y \in X$ such that $x \in^m A \circ B$ and $y \in^n A \circ B$, $xy \in_+^{(m \wedge n)} (A \circ B)$. Similarly, since $B \circ A$ is a multigroup under the same conditions, $yx \in_+^{(m \wedge n)} (B \circ A)$. Since $xy = yx$ for any $x, y \in X$ then $A \circ B = B \circ A$.
Conversely, assume $A \circ B = B \circ A$. If the multigroup product is commutative, then both multiplication and inverse conditions hold symmetrically for $A \circ B$. Thus, $A \circ B$ is closed under multiplication and inverse conditions. Hence, $A \circ B$ is a multigroup over X.

Proposition 7 Let A and B be multigroups over a group X. Then $A \cap B$ is a multigroup over X
Proof
For any $x \in^m (A \cap B)$ and $y \in^n (A \cap B)$ since $x \in^m A$and $x \in^m B$, and $y \in^n A$ and $y \in^n B$, by the multiplication condition for multigroups,

$$xy \in_+^{(m \wedge n)} A \text{and } xy \in_+^{(m \wedge n)} B \tag{5}$$

Thus:

$$xy \in_+^{(m \wedge n)} A \cap B \tag{6}$$

Hence, the multiplication condition holds. For any $x \in^m (A \cap B)$, since $x \in^m A$ and $x \in^m B$, by the inverse condition for multigroups, $x^{-1} \in_+^m A$ and $x^{-1} \in_+^m B$. Thus

$$x^{-1} \in_+^m (A \cap B) \tag{7}$$

Hence $A \cap B$ is a multigroup over X.
Next, we show the case is not true for union. Consider the cyclic group $X = \{e, a, b, c\}$ where e is the identity element and the group operation is as follows:
$a^2 = e, b^2 = e, c^2 = e,$
$ab = c, ac = b, bc = a,$
$ba = c, ca = b, cb = a,$
$ee = e, ea = a, eb, b, ec = c$
Consider the multigroups $A = \{e, a, a, b\}$ and $B = \{e, b, b, c\}$ over X. With the above information we are now ready to state and prove the next proposition.

Remark 1 It is worth noting here that $A \cup B$ is not necessarily a multigroup. To see this consider the multigroups A and B over the group X as in the above proposition. Consider their union $A \cup B = \{e, a, a, b, b, c\}$. For the union to be a multigroup it must satisfy the multiplication condition: For any $x \in^m (A \cup B)$ and $y \in^n (A \cup B)$,

$$xy \in_+^{(m \wedge n)} (A \cup B) \tag{8}$$

However, consider a and b from $A \cup B$. $a \in^2 (A \cup B)$ and $b \in^2 (A \cup B)$. However, $ab = c \in^1 (A \cup B)$. Hence,

$$ab \notin_+^{(m \wedge n)} (A \cup B) \tag{9}$$

Thus, the multiplication condition is not satisfied.

Definition 8 Let A and B be multigroups over a group X. Then A is a submultigroup of B denoted $A \subseteq B$if $m \leq n\ \forall x \in X$ such that $x \in^m A$ and $x \in^n B$.

Example 2 Let X be the cyclic group of order 4: $X = \{e, a, a^2, a^3\}$ under the multiplication modulo 4, where e is the identity element. Consider the multigroups $A = \{e, a, a^2, a^3, a^3\}$ and $B = \{e, a, a^2, a^3, a^3, a^3\}$. Since the multiplicity of every element in A is less or equal to that in B, then $A \subseteq B$. Moreover, B is a multigroup based on definition.

In Propositions 8 we present the results that establish the equivalence between the properties of multigroup based on conjugation of elements.

Proposition 8 Let A be a multiset over a set X. Then the following assertions are equivalent:

i. $xy \in^m A \wedge yx \in^n A \Longrightarrow m = n, \forall x, y \in X$
ii. $xyx^{-1} \in^m A \wedge y \in^n A \Longrightarrow m = n, \forall x, y \in X$

Proof:

To prove (i) $\Longrightarrow$ (ii):

Assume $xyx^{-1} \in^m A \wedge y \in^n A$. We know that $xyx^{-1} = (xy)x^{-1}$. Puttingyx^{-1} in place of y in (i) we get:

$$x(yx^{-1}) \in^m A \wedge (yx^{-1})x \in^n A \Longrightarrow m = n \quad (10)$$

This simplifies to

$$x(yx^{-1}) \in^m A \wedge y \in^n A \Longrightarrow m = n \quad (11)$$

To prove (ii) $\Longrightarrow$ (i): Assume $xyx^{-1} \in^m A \wedge y \in^n A \Longrightarrow m = n$. The statement essentially says that the multiplicity of an element $\boldsymbol{y}$ remains the same when conjugated by any element $\boldsymbol{x}$, i.e., $x(yx^{-1})$ has the same multiplicity as $\boldsymbol{y}$ in the multiset A. In particular, if xy and yx have the same multiplicity, it suggests that the multiset structure respects some form of symmetry under element swapping. Hence, $xy \in^m A \wedge yx \in^n A \Longrightarrow m = n, \forall x, y \in X$.

In Propositions 9 we present the results that establish the equivalence between the properties of multigroup based on product of their elements.

Proposition 9 Let A be a multiset over a set X. Then the following assertions are equivalent:

i. $A \circ B = B \circ A$, for all multisets B over X
ii. $xy \in^m A \wedge yx \in^n A \Longrightarrow m = n, \forall x, y \in X$

Proof:

(i) $\Longrightarrow$ (ii)

Assume $xy \in^m A \wedge yx \in^n A \Longrightarrow m = n, \forall x, y \in X$. Consider any multiset B over X

By the definition of the product of two multisets:

$$A \circ B = \{xy : x \in_+^p A, y \in_+^q B\}$$
$$B \circ A = \{yx : y \in_+^q B, x \in_+^p A\}$$

By hypothesis, the multiplicity of the product xy and yx is the same, it follows that:

$$A \circ B = B \circ A \quad (12)$$

(ii) $\Longrightarrow$ (i)

Assume $A \circ B = B \circ A$, for all multisets B over X. Choose an element $y \in B$ and form a singleton multiset $\{y\}$. Now

$$A \circ \{y\} = \{y\} \circ A \quad (13)$$

This means

$$\{xy : x \in_+^m A\} = \{yx : x \in_+^n A\} \quad (14)$$

Hence, $xy \in^m A \wedge yx \in^n A \Longrightarrow m = n, \forall x, y \in X$

CONCLUSION

In this study, we have extended classical group theory into the realm of multisets, introducing and formalizing the concept of multigroups. By utilizing Singh's dressed epsilon notation, which denotes that an element belongs to a multiset at least once, we have analyzed multisets and some$r$-level reference sets and their intrinsic subgroup properties.

Through the propositions and proofs presented, we established the foundational properties of multigroups, showing the conditions under which a multiset qualifies as a multigroup. Our exploration of multigroup operations, including multiplication, inversion, intersection, and union, revealed the nuanced ways in which these operations preserve or alter multigroup structures.

The incorporation of the dressed epsilon notation has been instrumental in extending the applicability of group theory to multisets, allowing for flexible expression multiplicities objects. This advancement not only bridges the gap between traditional group theory and multiset theory but also opens avenues for practical applications in areas where multiblicities of objects is significant. This work paves the way for future research into the applications of multigroups in computational mathematics, data analysis, and other related fields.

REFERENCE

Awolola, Johnson Aderemi (2019). On multiset relations and factor multigroups, *South East Asian J. of Mathematics and Mathematical Sciences*, 15(3), 1-10.

Awolola, Johnson Aderemi and Ejegwa (2017). On some algebraic properties of order of an element of a multigroup, *Quasigroups and Related Systems*, Vol. 25, 21-26.

Awolola, Johnson Aderemi and Ibrahim, Adeku Musa (2016). Some results on multigroups, *Quasigroups and Related Systems* Vol. 24, 169-177.

Baumslag, B., & Chandler, B. (1968). Theory and problems of group theory, Schaum's Outline Series.

Singh, D. (2006). *Multiset Theory: A New Paradigm of Science: an Inaugural Lecture*. University Organized Lectures Committee, Ahmadu Bello University.

Singh, D., Ibrahim, A. M., Yohanna, T., & Singh, J. N. (2008). *A systematization of fundamentals of multisets*. LecturasMatemáticas, 29, 33–48.

Singh, D., & Isah, A. I. (2016). Mathematics of Multisets: a unified Approach. AfrikaMatematika, 27(7), 1139-1146. Chicago

Ejegwa, P. A., & Ibrahim, A. M. (2020). Some properties of multigroups. *Palestine Journal of Mathematics*, *9*(1), 31-47.

Ibrahim, A. M., Awolola, J. A., & Alkali, A. J. (2016). An extension of the concept of n-level sets to multisets. *Annals of Fuzzy Mathematics and Informatics*, *11*(6), 855-862.

Nazmul, S. K. Majumdar, P. & Samanta S. K. (2013). On Multisets and Multigroups, *Ann. Fuzzy Math. Informa.* Vol. 6, No. 3, 643–656.

Peter, C., Balogun, F. & Adeyemi, O. A. (2024). an exploration of antimultigroup extensions. *fudma journal of sciences*, *8*(5), 269-273. *https://doi.org/10.33003/fjs-2024-0805-2719*