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Analysis of Equilibrium Points of Perturbed Circular Restricted Three-Body Problem (CRTBP) with Variable Mass.

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ABSTRACT
This study develops a dynamic mathematical equation that explicitly describes the motion of an infinitesimal mass with variable mass changes in the circular restricted three-body problem under the influence of perturbation factors such as radiation pressure due to the first oblate-radiating primary, albedo from the second oblate primary, oblateness, and a disc. The study looked at the impact of changing mass, radiation, albedo, oblateness, and disc characteristics on the existence and position of equilibrium points. The equation's dimensions were analyzed using Jean's law. We determine an adequate approximation for the locations of equilibrium points. Furthermore, various graphical investigations are provided to demonstrate the influence of parameters on point location. We discovered that these perturbations influence the positions of these equilibrium points. This discovery has numerous applications, particularly in the dynamical behaviour of tiny things like cosmic dust and grains.

INTRODUCTION

Many scientists investigated the restricted three body problem by considering many perturbations such as varied forms of the primary, solar radiation pressure, resonance, mass fluctuation, coriolis and centrifugal forces, the Yarkovsky effect, Poynting-Robertson drags, the Albedo effect, etc. (Szebehely, 1967; El-Shaboury and Mustafa, 2013; Abouelmagd and El-Shaboury, 2012). The three-body problem (N = 3) has been thoroughly explored throughout the last three centuries. A few closed form solutions have been found under particular assumptions. Unlike the two-body issue, the three-body problem lacks a simple generic solution that can be written down to describe the motion over time for any arbitrary initial conditions.

The purpose of this study is to look into the effects of albedo, dust disk, and variable mass on the location of the equilibrium points in the circular restricted three-body problem. The study of the circular restricted three-body problem has generated numerous novel ideas (for example, Poincare sections) and influenced other fields of mathematics and mechanics, such as topology. It also improves knowledge and understanding of the current mathematical methodologies in use. Now the search is on for planets orbiting distant stars, and we return to a centuries-old problem: determining where these extra-solar planets could exist.

As a result, further theoretical research is required in order to broaden our understanding of the potential solutions available to orbit designers. Radwan and Nihad (2021) assessed the altered positions and the linear stability of the triangular points within the framework of the elliptic restricted three-body problem, treating the two primaries as triaxial structures. The positions of these points were discovered to be influenced by the triaxiality coefficients of the primaries and the eccentricity of their orbits. Gyegwa, et al (2025) investigate the effect zonal hamonics on the motion of a satellite by applying a generalized R3BP to the EQ pegasi binary system and using Lyapunov stability theorem; the collinear equilibrium points remain unstable. Abdullah et al (2021) investigated the dynamical evolution of a variable mass that is infinitesimal, influenced by Newtonian and Yukawa potentials in the circular restricted three-body system, assuming that the infinitesimal body changes its mass according to Jeans law. Five equilibrium points similar to those in the classical restricted three-body problem have been discovered. However, Yukawa parameters have minimal impact on the model, while variation parameters significantly influence the positioning of equilibrium points.
From literature, it is noted that certain authors examined a scenario where two primary bodies are oblate spheroids surrounded by a dust disc in space with mass;

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the larger oblate primary acts as a radiating source, while the second oblate primary exhibits an albedo effect, though variable mass was overlooked. Although some authors examined the influence of albedo on the movement of an infinitesimal body within the circular restricted three-body problem by varying all masses, they overlooked a disk of dust in space surrounding the masses. In this study, we performed research on a perturbed circular restricted three-body problem with variable mass under the conditions that (i) both primaries are oblate spheroids surrounded by a dust disc in space with mass, (ii) the larger oblate primary acts as a radiating body, (iii) the smaller oblate primary exhibits an albedo effect, and (iv) the infinitesimal body has a variable mass. We apply Jeans transformation law to derive the motion equation of the infinitesimal primary, thereby building upon the research of Abdullah, et al (2017), Akere-Jaiyeola, et al (2019), and Radwan and Nihad (2021).

MATERIALS AND METHODS

For variable-mass systems, Newton's second law of motion cannot be directly implemented since it strictly applies to systems with constant mass (Plastino and Muzio, 1992). To analyse bodies with time-varying mass, we must modify Newton's second law by incorporating an additional term that accounts for the momentum transfer associated with mass entering or exiting the system. This reformulation yields the general equation of motion for variable-mass system and in the framework of the loss of mass being taken non-isotropic, the equations of motion for the infinitesimal body in the inertial frame in the case that the escaping or incoming mass occurs from n points has the form:

$$m_3\frac{d^2\vec{r}}{dt^2} = m_3\frac{\partial^2\vec{r}}{\partial t^2} + 2m_3\vec{\omega}\times\dot{\vec{r}} + m_3\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\frac{Gm_3m_1}{\rho_1^3}(\vec{r}-\vec{\rho}_1) - \frac{Gm_3m_2}{\rho_2^3}(\vec{r}-\vec{\rho}_2) - \dot{m}_3\sum_{i=1}^{n}\vec{u}_i \quad (1)$$

The last term in equation (1), will vanish in two cases: when the value of the sum equals zero or $\vec{v}_i = \dot{\vec{r}}_3$. Consequently, the loss of mass is isotropic in the two cases.

Set m_1 as the first oblate primary, m_2 the second oblate primary, and F_r and F_a are the radiation pressure force, mass reduction factor q, and albedo Q_A then;

$$q = \left(1 - \frac{F_r}{F_{g1}}\right) \text{ and } Q_A = \left(1 - \frac{F_a}{F_{g2}}\right), \quad (2)$$

where F_{g1} and F_{g2} are gravitational forces of the respective primaries. Let A_1 and A_2 be the oblateness coefficients of first and second primaries, which are respectively defined (McCuskey, 1963; Abouelmagd and Sharaf, 2013) as:

$$A_1 = \frac{R_{e1}^2 - R_{p1}^2}{5R^2}, \text{ and } A_2 = \frac{R_{e2}^2 - R_{p2}^2}{5R^2}, \quad (3)$$

Where R_{ei} and R_{pi} for $i = 1, 2$ are equatorial and polar radii of the primaries and R is the separation between both the primaries.

The gravitational force exerted due to disc of the mass M_d on the infinitesimal mass defines a potential, which is expressed (Miyamoto and Nagai, 1975; Kushvah, 2008) as:

$$\phi(r,0) = \frac{M_d}{\sqrt{r^2+T^2}} \text{ where } r = \sqrt{x^2+y^2} \text{ and } T = (a^* + b^*) \quad (4)$$

measures the density profile of the disc. M_d is the total mass of the disc, r is the radial distance of the infinitesimal body, while r_c is that in the classical case, T is the sum of flatness parameter a^* and core parameter b^*.

The forces acting on m due to m_1 and m_2 are $F_{g1}q$ and $F_{g2}Q_A$ respectively and having the coordinates $(\mu R, 0, 0)$ and $(-(1-\mu)R, 0, 0)$ respectively too and $(x, y, 0)$ be that of infinitesimal mass in the $xy-$plane, by the definition of the centre of mass,

$$\rho_1 = \frac{m_2}{m_1+m_2}R, \qquad \rho_2 = \frac{m_1}{m_1+m_2}R = \left(1 - \frac{m_2}{m_1+m_2}\right)R \quad (5)$$

$$\text{Mass ratio } \mu = \frac{m_2}{m_1+m_2} \quad (6)$$

The equations of motion of the infinitesimal mass of variable mass can be written as:

$$\ddot{\vec{r}} + 2\vec{\omega}\times\dot{\vec{r}} + n\frac{\dot{m}}{m}\dot{\vec{r}} = -\frac{Gm_1q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1^3}(\vec{r}-\vec{\rho}_1) - \frac{Gm_2Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2^3}(\vec{r}-\vec{\rho}_2) - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \frac{M_d\vec{r}}{m(r^2+T^2)^{\frac{3}{2}}} \tag{7}$$

where $\vec{\omega} = (0,\ 0,\ \omega)$ and

$$\rho_1^2 = (x-\mu R)^2 + y^2\ , \qquad \rho_2^2 = (x+(1-\mu)R)^2 + y^2$$

If the rotation frames rotate with the angular velocity ω , the relation between the inertial and rotating coordinates is governed by the x –coordinate of

$$n\frac{\dot{m}}{m}\dot{\vec{r}} = n\frac{\dot{m}}{m}(\dot{x}-\omega y) \tag{8}$$

and the y –coordinate of

$$n\frac{\dot{m}}{m}\dot{\vec{r}} = n\frac{\dot{m}}{m}(\dot{y}+\omega x) \tag{9}$$

Then equations (7) can be expressed as:

$$\ddot{\vec{r}} - 2\omega(\dot{y},-\dot{x}) + n\frac{\dot{m}}{m}(\dot{x}-\omega y, \dot{y}+\omega x) = -\frac{Gm_1q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1^3}(\vec{r}-\vec{\rho}_1) - \frac{Gm_2Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2^3}(\vec{r}-\vec{\rho}_2) - \frac{M_d\vec{r}}{m(r^2+T^2)^{\frac{3}{2}}} + \omega^2\vec{r} \tag{10}$$

Here, the second term on the left-hand side and last term on the right-hand side of equation (10) are the Coriolis acceleration and the centrifugal acceleration respectively. The components of equation (10) reduce to

$$\ddot{x} - 2\omega\dot{y} + n\frac{\dot{m}}{m}(\dot{x}-\omega y) = -\frac{Gm_1q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1^3}(x-\mu R) - \frac{Gm_2Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2^3}(x+(1-\mu)R) - \frac{M_dx}{m(r^2+T^2)^{\frac{3}{2}}} + \omega^2x \tag{11}$$

$$\ddot{y} + 2\omega\dot{x} + n\frac{\dot{m}}{m}(\dot{y}+\omega x) = -\frac{Gm_1q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1^3}y - \frac{Gm_2Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2^3}y - \frac{M_dy}{m(r^2+T^2)^{\frac{3}{2}}} + \omega^2y \tag{12}$$

Equations (11) and (12) can be written in compact form as:

$$\ddot{x} - 2\omega\dot{y} + n\frac{\dot{m}}{m}(\dot{x}-\omega y) = \frac{\partial U}{\partial x} \tag{13}$$

$$\ddot{y} + 2\omega\dot{x} + n\frac{\dot{m}}{m}(\dot{y}+\omega x) = \frac{\partial U}{\partial y} \tag{14}$$

And

$$U(x,y) = \frac{\omega^2}{2}(x^2+y^2) + \frac{Gm_1q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1} + \frac{Gm_2Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2} + \frac{2M_d}{m\sqrt{r^2+T^2}} \tag{15}$$

(13) and (14) is the sum of the gravitational and centrifugal potential and belt.
Applying Jeans' space-time transformation law

With $x = R\eta^{-q}x',\quad y = R\eta^{-q}y',\quad t = \eta^{-k}\tau,\ \eta = \dfrac{m}{m_0}$ (16)

The mass of the third body at time $t=0$ is m_0. Where $s=1, k=0, q=\frac{1}{2}$.

Therefore,

$$\frac{d\eta}{dt}=-\sigma\eta,\quad \frac{\dot{m}}{m}=-\sigma,\quad x=R\eta^{-\frac{1}{2}}x',\quad y=R\eta^{-\frac{1}{2}}y',\quad dt=d\tau,\quad U=R^2\eta^{-1}\Omega \tag{17}$$

From (17), the velocity and acceleration components becomes;

$$\dot{x}=R\eta^{-\frac{1}{2}}\dot{x}'+Rx'\frac{d\eta^{-\frac{1}{2}}}{d\eta}\frac{d\eta}{dt}=\eta^{-\frac{1}{2}}R\dot{x}'+Rx'\left(-\frac{1}{2}\eta^{-\frac{3}{2}}\right)(-\sigma\eta)=R\eta^{-\frac{1}{2}}\left(\dot{x}'+\frac{1}{2}\sigma x'\right) \tag{18}$$

$$\dot{y}=R\eta^{-\frac{1}{2}}\dot{y}'+Ry'\frac{d\eta^{-\frac{1}{2}}}{d\eta}\frac{d\eta}{dt}=\eta^{-\frac{1}{2}}R\dot{y}'+Ry'\left(-\frac{1}{2}\eta^{-\frac{3}{2}}\right)(-\sigma\eta)=R\eta^{-\frac{1}{2}}\left(\dot{y}'+\frac{1}{2}\sigma y'\right) \tag{19}$$

$$\ddot{x}=R\eta^{-\frac{1}{2}}\left(\ddot{x}'+\sigma\dot{x}'+\frac{1}{4}\sigma^2x'\right) \tag{20}$$

$$\ddot{y}=R\eta^{-\frac{1}{2}}\left(\ddot{y}'+\sigma\dot{y}'+\frac{1}{4}\sigma^2y'\right) \tag{21}$$

Using the Jeans transformation; (11) and (12) becomes

$$\frac{R}{\sqrt{\eta}}\left(\frac{d^2x'}{dt^2}+\sigma\frac{dx'}{dt}+\frac{1}{4}\sigma^2x'\right)-2\omega\frac{R}{\sqrt{\eta}}\left(\frac{dy'}{dt}+\frac{1}{2}\sigma y'\right)-n\sigma\frac{R}{\sqrt{\eta}}\left(\frac{dx'}{dt}+\frac{1}{2}\sigma x'\right)+$$

$$n\sigma\omega\frac{R}{\sqrt{\eta}}y'=\frac{\frac{R^2}{\eta}}{\frac{R}{\sqrt{\eta}}}\frac{\partial\Omega}{\partial x'}=-\frac{Gm_1q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1^3}\frac{R}{\sqrt{\eta}}\left(x'-\mu\sqrt{\eta}\right)- \tag{22}$$

$$\frac{Gm_2Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2^3}\frac{R}{\sqrt{\eta}}\left(x'+(1-\mu)\sqrt{\eta}\right)-\frac{M_d'\frac{R}{\sqrt{\eta}}x'}{\left(r'^2+T'^2\right)^{\frac{3}{2}}}+\omega^2\frac{R}{\sqrt{\eta}}x'$$

$$\frac{R}{\sqrt{\eta}}\left(\frac{d^2y'}{dt^2}+\sigma\frac{dy'}{dt}+\frac{1}{4}\sigma^2y'\right)+2\omega\frac{R}{\sqrt{\eta}}\left(\frac{dx'}{dt}+\frac{1}{2}\sigma x'\right)-n\sigma\frac{R}{\sqrt{\eta}}\left(\frac{dy'}{dt}+\frac{1}{2}\sigma y'\right)-$$

$$n\sigma\omega\frac{R}{\sqrt{\eta}}x'=\frac{\frac{R^2}{\eta}}{\frac{R}{\sqrt{\eta}}}\frac{\partial\Omega}{\partial y'}=-\frac{Gm_1q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1^3}\frac{R}{\sqrt{\eta}}y'-\frac{Gm_2Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2^3}\frac{R}{\sqrt{\eta}}y'- \tag{23}$$

$$\frac{M_d'\frac{R}{\sqrt{\eta}}y'}{\left(r'^2+T'^2\right)^{\frac{1}{2}}}+\omega^2\frac{R}{\sqrt{\eta}}y'$$

Dimensionless parameters ρ_1 and ρ_2 are expressed as following:

$$\rho_1=\frac{R}{\sqrt{\eta}}\left(\left(x'-\mu\sqrt{\eta}\right)^2+y'^2\right)^{\frac{1}{2}}=\frac{R}{\sqrt{\eta}}r_1 \tag{24}$$

$$\rho_2 = \frac{R}{\sqrt{\eta}}\left(\left(x' + (1-\mu)\sqrt{\eta}\right)^2 + y'^2\right)^{\frac{1}{2}} = \frac{R}{\sqrt{\eta}} r_2 \tag{25}$$

$$r_1 = \left(\left(x' - \mu\sqrt{\eta}\right)^2 + y'^2\right)^{\frac{1}{2}}, \qquad r_2 = \left(\left(x' + (1-\mu)\sqrt{\eta}\right)^2 + y'^2\right)^{\frac{1}{2}}$$

Noticing that

$$\frac{Gm_1}{\omega^2 R^3} = \frac{m_1}{m_1 + m_2} = 1-\mu \tag{26}$$

and

$$\frac{Gm_2}{\omega^2 R^3} = \frac{m_2}{m_1 + m_2} = \mu \tag{27}$$

Since

$$\omega^2 = G\frac{m_1 + m_2}{R^3} \tag{28}$$

and we arrive at the following equations.

$$\frac{d^2x}{dt^2} - 2\omega\frac{dy}{dt} - \sigma(n-1)\frac{dx}{dt} + \sigma\omega(n-1)y = \frac{\partial\Omega}{\partial x} = -\frac{q\left(\sqrt{\eta}\right)^3\left(1+\frac{\eta A_1}{2r_1^2}\right)}{r_1^3}(1-\mu)\left(x - \mu\sqrt{\eta}\right) - \frac{Q_A\left(\sqrt{\eta}\right)^3\left(1+\frac{\eta A_2}{2r_2^2}\right)}{r_2^3}\mu\left(x + (1-\mu)\sqrt{\eta}\right) - \frac{M_d x}{\left(r^2 + T^2\right)^{\frac{3}{2}}} + \omega^2 x + \frac{\sigma^2(2n-1)}{4}x \tag{29}$$

$$\frac{d^2y}{dt^2} + 2\omega\frac{dx}{dt} - \sigma(n-1)\frac{dy}{dt} - \sigma\omega(n-1)x = \frac{\partial\Omega}{\partial y} = -\frac{q\left(\sqrt{\eta}\right)^3\left(1+\frac{\eta A_1}{2r_1^2}\right)}{r_1^3}(1-\mu)y - \frac{Q_A\left(\sqrt{\eta}\right)^3\left(1+\frac{\eta A_2}{2r_2^2}\right)}{r_2^3}\mu y - \frac{M_d y}{(r^2+T^2)^{\frac{3}{2}}} + \omega^2 y + \frac{\sigma^2(2n-1)}{4}y \tag{30}$$

Equations (29) and (30) can be rewritten in a compact form as

$$\ddot{x} - 2\omega\dot{y} - \sigma(n-1)\dot{x} + \sigma\omega(n-1)y = \frac{\partial\Omega}{\partial x} \tag{31}$$

$$\ddot{y} + 2\omega\dot{x} - \sigma(n-1)\dot{y} - \sigma\omega(n-1)x = \frac{\partial\Omega}{\partial y} \tag{32}$$

and

$$\Omega(x,y) = \left(\frac{\omega^2}{2} + \frac{\sigma^2(2n-1)}{8}\right)\left((1-\mu)r_1^2 + \mu r_2^2 - \eta\mu(1-\mu)\right) + \left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\frac{\eta A_1}{2r_1^2}\right)}{r_1} + \frac{Q_A\mu\left(1+\frac{\eta A_2}{2r_2^2}\right)}{r_2}\right) + \frac{2M_d}{(r^2+T^2)^{\frac{1}{2}}} \tag{33}$$

Since

$$(1-\mu)r_1^2 + \mu r_2^2 - \eta\mu(1-\mu) = x^2 + y^2 \tag{34}$$

The perturbed mean ω is given as:

$$\omega = \sqrt{1 + \frac{3}{2}(A_1 + A_2) + \frac{2M_d r_c}{\left(r_c^2 + T^2\right)^{\frac{3}{2}}}} \tag{35}$$

Then, we have

$$\omega^2 = 1 + \frac{3}{2}(A_1 + A_2) + \frac{2M_d r_c}{\left(r_c^2 + T^2\right)^{\frac{3}{2}}} \tag{36}$$

These can be rewritten for ($n = 1$) in the form

$$\ddot{x} - 2\omega\dot{y} = \frac{\partial\Omega}{\partial x} \tag{37}$$

$$\ddot{y} + 2\omega\dot{x} = \frac{\partial\Omega}{\partial y} \tag{38}$$

And, $$\Omega(x,y)=\left(\frac{\omega^2}{2}+\frac{\sigma^2}{8}\right)\left((1-\mu)r_1^2+\mu r_2^2-\eta\mu(1-\mu)\right)+\left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\frac{\eta A_1}{2r_1^2}\right)}{r_1}+\frac{Q_A\mu\left(1+\frac{\eta A_2}{2r_2^2}\right)}{r_2}\right)+\frac{2M_d}{(r^2+T^2)^{\frac{1}{2}}} \tag{39}$$

$$\frac{\partial\Omega}{\partial x}=\omega^2x+\frac{\sigma^2}{4}x-\left(\sqrt{\eta}\right)^3\left(\begin{array}{l}\frac{q(1-\mu)\left(1+\frac{3\eta A_1}{2r_1^2}\right)}{r_1^3}\left(x-\sqrt{\eta}\mu\right)+\\ \frac{Q_A\mu\left(1+\frac{3\eta A_2}{2r_2^2}\right)}{r_2^3}\left(x+\sqrt{\eta}(1-\mu)\right)\end{array}\right)-\frac{M_d x}{\left(r^2+T^2\right)^{\frac{3}{2}}} \tag{40}$$

and

$$\frac{\partial\Omega}{\partial y}=\omega^2y+\frac{\sigma^2}{4}y-\left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\frac{3\eta A_1}{2r_1^2}\right)}{r_1^3}y+\frac{Q_A\mu\left(1+\frac{3\eta A_2}{2r_2^2}\right)}{r_2^3}y\right)-\frac{M_d y}{\left(r^2+T^2\right)^{\frac{3}{2}}} \tag{41}$$

Equations (37) and (38) are different from the classical equations by the extra terms $\frac{\sigma^2}{4}x$ and $\frac{\sigma^2}{4}y$ due to the variation in the mass of the third body.

Locations of Equilibrium (Libration) Points

In the synodic reference system, there exist five equilibrium points; called Lagrange or libration points L_i $(i=1,\ldots 5)$, which can be computed by imposing velocity $\{\dot{x},\dot{y}\}$ and acceleration $\{\ddot{x},\ddot{y}\}$ to be null in the equations of motion (35) and (36).
This is,

$$0=\omega^2x+\frac{\sigma^2}{4}x-\left(\sqrt{\eta}\right)^3\left(\begin{array}{l}\frac{q(1-\mu)\left(1+\frac{3\eta A_1}{2r_1^2}\right)}{r_1^3}\left(x-\sqrt{\eta}\mu\right)+\\ \frac{Q_A\mu\left(1+\frac{3\eta A_2}{2r_2^2}\right)}{r_2^3}\left(x+\sqrt{\eta}(1-\mu)\right)\end{array}\right)-\frac{M_d x}{\left(r^2+T^2\right)^{\frac{3}{2}}} \tag{42}$$

$$0=\left(\omega^2+\frac{\sigma^2}{4}-\left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\frac{3\eta A_1}{2r_1^2}\right)}{r_1^3}+\frac{Q_A\mu\left(1+\frac{3\eta A_2}{2r_2^2}\right)}{r_2^3}\right)-\frac{M_d}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right)y \tag{43}$$

The second equation (43), is verified in two cases:

1. If $y=0$;
2. If $y\neq 0$ and

$$\left(\omega^2+\frac{\sigma^2}{4}-\left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\frac{3\eta A_1}{2r_1^2}\right)}{r_1^3}+\frac{Q_A\mu\left(1+\frac{3\eta A_2}{2r_2^2}\right)}{r_2^3}\right)-\frac{M_d}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right)=0$$

The first condition corresponds to the *collinear equilibrium points* L_1, L_2 and L_3, the second to the *triangular equilibrium points* L_4 and L_5.

Collinear Equilibrium Points

Let us introduce the variable u as

$u=x+\sqrt{\eta}(1-\mu)$, Then we have

$$\left(\omega^2+\frac{\sigma^2}{4}-\frac{M_d}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right)\left(u-\sqrt{\eta}(1-\mu)\right)-\left(\sqrt{\eta}\right)^3\left(\begin{array}{c}\frac{q(1-\mu)\left(1+\frac{3\eta A_1}{2\left(u-\sqrt{\eta}\right)^2}\right)}{\left(u-\sqrt{\eta}\right)^2}s_1+\\ \frac{Q_A\mu\left(1+\frac{3\eta A_2}{2(u)^2}\right)}{(u)^2}s_0\end{array}\right)=0, \qquad (44)$$

Where $s_0=\operatorname{sgn}(u)$ and $s_1=\operatorname{sgn}\left(u-\sqrt{\eta}\right)$.

We can distinguish among different combinations of s_0 and s_1 as a function of the position of the collinear point, by identifying three intervals on the x - axis, namely,

$L_3:\quad (s_0,s_1)=(-1,-1)$ if $x<\sqrt{\eta}\mu$,

$L_1:\quad (s_0,s_1)=(-1,1)$ if $\sqrt{\eta}\mu<x<-\sqrt{\eta}(1-\mu)$,

$L_2:\quad (s_0,s_1)=(1,1)$ if $x>-\sqrt{\eta}(1-\mu)$.

Using series expansion; each of these three equations admits one real solution and two pair of conjugate complex solutions and thus we have three collinear equilibrium points as:

$$L_1=x_1=\sqrt{\eta}(\mu-1)+u \qquad (45) \qquad\qquad L_2=x_2=\sqrt{\eta}(\mu-1)+u \qquad (46)$$

$$L_3=x_3=\sqrt{\eta}(\mu-1)+u \qquad (47)$$

Location of triangular equilibrium points

The triangular points L_4 and L_5, also belong to the xy plane and are given by the solution of equations

$$\Omega_x=0=\Omega_y,\qquad y\neq 0 \qquad (48).$$

Note that from (37), we have

$$\frac{\partial\Omega}{\partial x}=\left(x-\sqrt{\eta}\mu\right)f(r_1)+\left(x+\sqrt{\eta}(1-\mu)\right)g(r_2) \qquad (49),$$

and

$$\frac{\partial}{\partial x}(r_1)=\frac{\left(x-\sqrt{\eta}\mu\right)}{r_1},\quad \frac{\partial}{\partial x}(r_2)=\frac{\left(x+\sqrt{\eta}(1-\mu)\right)}{r_2},\quad \frac{\partial}{\partial y}(r_1)=\frac{y}{r_1},\quad \frac{\partial}{\partial y}(r_2)=\frac{y}{r_2}$$

Thus equations (48) will satisfy if $f(r_1)=0=g(r_2)$, so that at equilibria points let $r_1=d_1$, $r_2=d_2$ and $(x,y)=(a,b)$, that is

$$0=\left(\omega^2+\frac{\sigma^2}{4}-\frac{M_d}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right)a-\left(\sqrt{\eta}\right)^3\left(\begin{array}{l}\dfrac{q(1-\mu)\left(1+\dfrac{3\eta A_1}{2d_1^2}\right)}{d_1^3}\left(a-\sqrt{\eta}\mu\right)+\\ \dfrac{Q_A\mu\left(1+\dfrac{3\eta A_2}{2d_2^2}\right)}{d_2^3}\left(a+\sqrt{\eta}(1-\mu)\right)\end{array}\right) \tag{50}$$

$$0=b\left(\left(\omega^2+\frac{\sigma^2}{4}-\frac{M_d}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right)-\left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\dfrac{3\eta A_1}{2d_1^2}\right)}{d_1^3}+\frac{Q_A\mu\left(1+\dfrac{3\eta A_2}{2d_2^2}\right)}{d_2^3}\right)\right) \tag{51}$$

The triangular points are the solutions of equations (10) and (11) when $b\neq 0$.
this is,

$$\sqrt{\eta}(1-\mu)\left(\left(\omega^2+\frac{\sigma^2}{4}-\frac{M_d}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right)-\frac{\left(\sqrt{\eta}\right)^3 q\left(1+\dfrac{3\eta A_1}{2d_1^2}\right)}{d_1^3}\right)=0 \tag{52}$$

and,

$$\sqrt{\eta}\mu\left(\left(\omega^2+\frac{\sigma^2}{4}-\frac{M_d}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right)-\frac{\left(\sqrt{\eta}\right)^3 Q_A\left(1+\dfrac{3\eta A_2}{2d_2^2}\right)}{d_2^3}\right)=0 \tag{53}$$

Now, if the bigger primary is neither oblate nor radiating, the smaller primary is neither oblate nor albedo, the infinitesimal body has no variable mass and there is no potential from a belt, then $\omega=1,\ A_i(i=1,2)=0,\ M_d=0,\ \sigma=0,\ q=1,\ Q_A=1,\ \eta=1$. Consequently, equations (50) and (51) reduces to

$$d_1^3=1 \text{ and } d_2^3=1 \Rightarrow \quad d_1=d_2=1 \tag{54}$$

Now, if we assume that the primaries are oblate, with the larger primary being a radiating body and the smaller one having albedo effects, while the infinitesimal body possesses variable mass and is influenced by an external belt potential, then $d_i\ (i=1,2)$ will take the form,

$$d_i=1+\delta_i,\quad |\delta_i|<<1,\qquad (i=1,2) \tag{55}$$

we obtain the triangular equilibrium points $L_4(a,b)$ and $L_5(a,-b)$ as:

$$\left.\begin{array}{c} a=\dfrac{1}{2}\left(\sqrt{\eta}(2\mu-1)+\dfrac{2}{\sqrt{\eta}}(\delta_2-\delta_1)\right)\\ b=\pm\dfrac{1}{2}\sqrt{(4-\eta)+4(\delta_1+\delta_2)}=\pm\dfrac{1}{2}\sqrt{(4-\eta)}\left(1+\dfrac{2}{(4-\eta)}(\delta_1+\delta_2)\right)\end{array}\right\} \tag{56}$$

Substituting the assumption (55), a and b into equations (50) and (51) we obtain

$$\delta_1 = \frac{1}{3\eta\sqrt{\eta}}\left(\begin{array}{l}\left(\frac{4\eta\sqrt{\eta}-5}{4}\right)-\left(\frac{12-25\eta+8\eta^2\sqrt{\eta}}{8}\right)A_1-\frac{3}{2}A_2+\frac{1}{2}p_3- \\ \frac{M_d\left(2r_c-1\right)}{\left(r_c^2+T^2\right)^{\frac{3}{2}}}-\frac{5}{4}p_1\end{array}\right) \tag{57}$$

$$\delta_2 = \frac{1}{3\eta\sqrt{\eta}}\left(\begin{array}{l}\left(\frac{4\eta\sqrt{\eta}-5}{4}\right)-\frac{3}{2}A_1-\left(\frac{12-25\eta+8\eta^2\sqrt{\eta}}{8}\right)A_2+\frac{1}{2}p_3- \\ \frac{M_d\left(2r_c-1\right)}{\left(r_c^2+T^2\right)^{\frac{3}{2}}}-\frac{5}{4}p_2\end{array}\right) \tag{58}$$

The coordinates of triangular equilibrium points a and b, by virtue of the two triangles they form with the line joining the primaries and are denoted by $L_{4,5}\left(a,\pm b\right)$.

RESULTS AND DISCUSSION

A mathematical equation describing the movement of infinitesimal masses with variable masses orbiting oblate primaries, one emitting radiation and the other producing an albedo effect, was created and tested. The purpose of this study was to assess the impact of disturbances on the positions of the libration points. We evaluated the presence and position of the equilibrium points when the perturbation parameters change. Maple 2015 version was used for the numerical simulations of mathematical models and are done with a variety of parameter settings. The effects of the involved parameters on the position of the equilibrium points are that the locations of the collinear points with respect to mass parameter μ from 0 to 0.5 for different values of σ and M_d, this shows that the collinear point L1 approaches the primaries as the variable mass parameter increases. However, at lower values of the mass parameter, L1 shifts away from the primaries, while for values above 0.4, it moves closer to them.

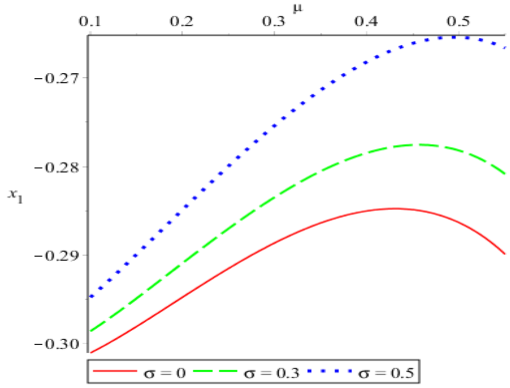

Figure 1: Graph Showing the Variation of Location of L1 versus μ.

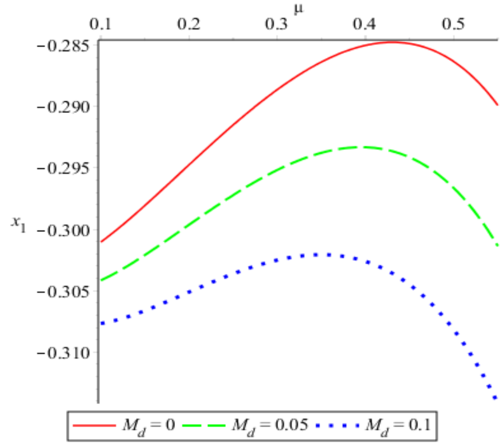

Figure 2: Graph Showing the Variation of Location of L1 versus μ.

L2 come nearer to the more massive primary with the increase in variable mass parameter and L3 moves away from the primaries with the increase in variable mass parameter. It is noted that the collinear point L1 move away from the primaries with the increase in disc parameter values while at smaller values of disc parameter, the location moves away from the primaries, but moves towards the primaries for the values of μ above 0.4. It is seen that L2 move away from the more massive primary with the increase in disc parameter values. L3 moves closer to the primaries with the increase in disc parameter values.

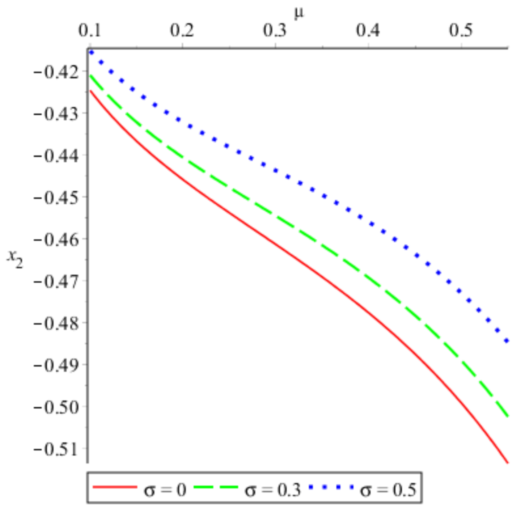

Figure 3: Graph Showing the Variation of Location of L2 versus μ.

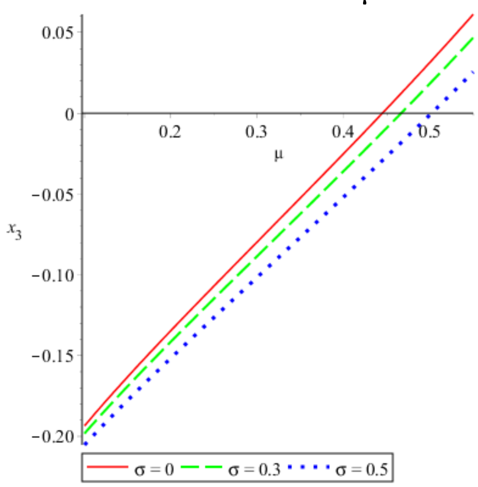

Figure 4: Graph Showing the Variation of Location of L3 versus μ.

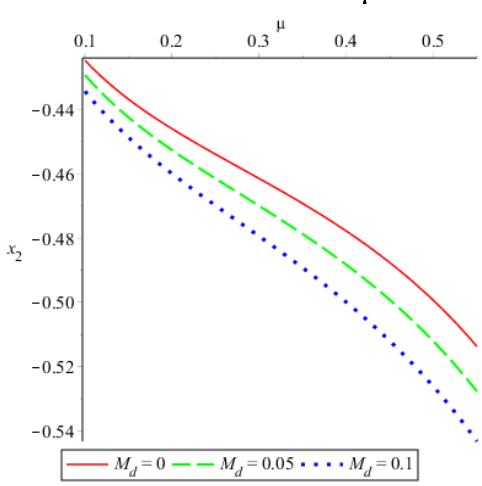

Figure 5: Graph Showing the Variation of Location of L2 versus μ.

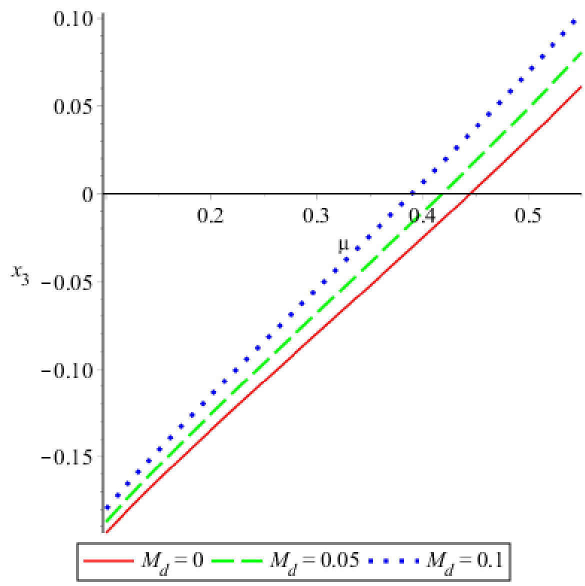

Figure 6: Graph Showing the Variation of Location of L3 versus μ.

The variation of triangular points $r_{L_{4,5}}$ against the mass ratio μ from 0 to 0.5 for different values of σ. It is clear from the figure that the locations begin to decrease slowly as mass ratio increases from 0 to 0.5. We also observe that the variation in the locations $r_{L_{4,5}}$ increases as variable mass parameter value increases. The variation of $r_{L_{4,5}}$ against the mass ratio μ from 0 to 0.5 for different values of q. It is clear from the figure that the locations begin to decrease sharply as mass ratio increases from 0 to 0.5. We also observe that the variation in the locations $r_{L_{4,5}}$ decreases as radiation pressure values increases. The variation of $r_{L_{4,5}}$ against the mass ratio μ from 0 to 0.5 for different values of Q_A. It is clear from the figure that the locations begin to decrease sharply as mass ratio increases from 0 to 0.5. We also observe that the variation in the locations $r_{L_{4,5}}$ increases as albedo parameter values increases. The variation of $r_{L_{4,5}}$ against the mass ratio μ from 0 to 0.5 for different values of M_d. It is clear from the figure that the locations begin to decrease slowly as mass ratio increases from 0 to 0.5. We also observe that the variation in the locations $r_{L_{4,5}}$ increases as disc parameter value increases.

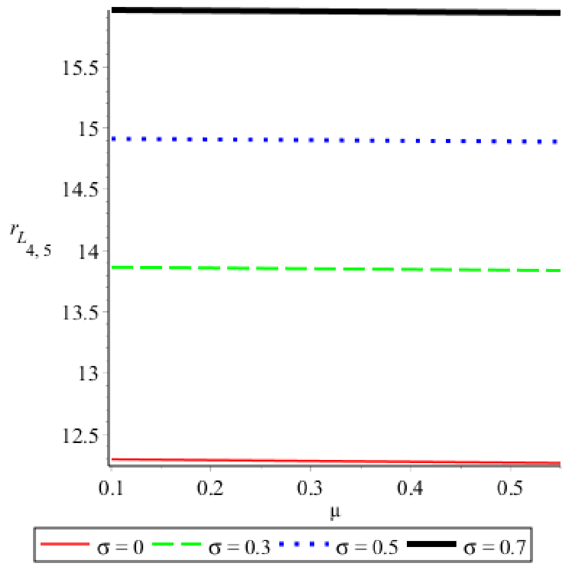

Figure 7: The location of $L_{4,5}$ with different values of variable mass parameter σ .

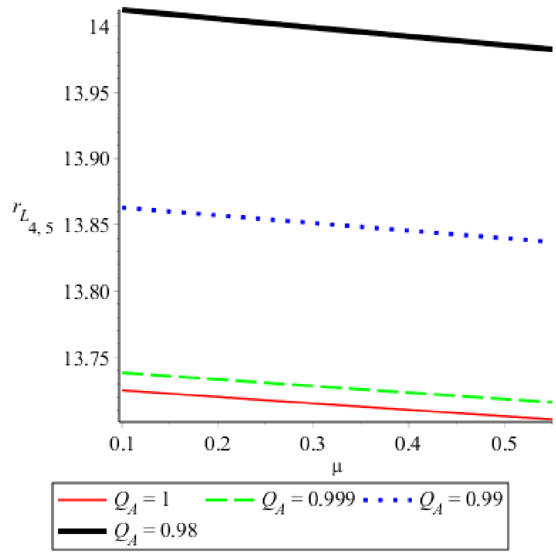

Figure 9: The location of $L_{4,5}$ with different values of albedo coefficient Q_A .

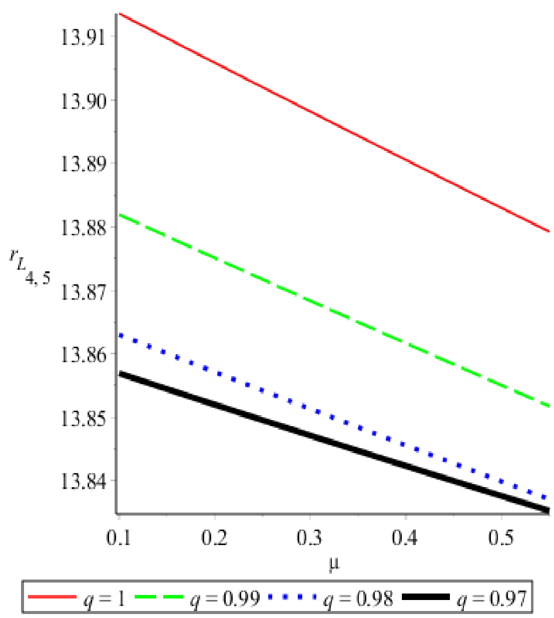

Figure 8: The location of $L_{4,5}$ with different values of radiation pressure q .

CONCLUSION

The circular restricted three-body problem is investigated in the setting of an infinitesimal body with changeable mass variations based on Jeans' law. The equation of motion is obtained when mass loss is not isotropic. We investigated the existence of both collinear and triangular points under the effect of perturbations in the form of radiation caused by an oblate-radiating first primary, the albedo of an oblate second primary, the oblateness of both primaries, the presence of a disc, and variable mass. In addition, a suitable estimate for the positions of collinear and triangular equilibrium points is found. Some graphical examinations into the parametric impacts of the variable mass on the placement of both collinear and triangular points are conducted. We found that the locations of these equilibrium points shifted from the original point because of these perturbations.

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