Journal of Basics and Applied Sciences Research (JOBASR)
ISSN (print): 3026-9091, ISSN (online): 1597-9962
Volume 4(1) January 2026
DOI: https://dx.doi.org/10.4314/jobasr.v4i1.3

A New Four Parameter Hybrid of Dhillon-Gompertz lifetime distribution with Bathtub-shaped Failure Rate: Properties and Applications

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ABSTRACT
This article introduced new four parameter lifetime distribution named Hybrid of Dhillon-Gompertz (HDG) using an additive methodology. Key properties of the model, including the cumulative distribution function, probability density function, failure rate, survival function, quantile function, mode, asymptotic behavior, order statistics, characteristic function moment and moment generating function, were derived. The parameters of the proposed distribution were estimated using the maximum likelihood estimation (MLE) method, and their applicability was tested using two sets of lifetime data. Goodness-of-fit comparison, based on Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Corrected Akaike Information Criterion (AICc), demonstrated the effectiveness of the proposed model. The findings indicate the HDG distribution exhibited superior performance compared to some existing distributions. These results highlight the potential of the proposed distributions to model lifetime data more accurately than existing alternatives. The findings of this study have significant theoretical and practical implications for reliability analysis and lifetime data modeling. The proposed hybrid distribution demonstrated superior performance over existing models, suggesting their potential for more accurate failure rate estimation and risk assessment in engineering, biomedical, and industrial applications.

Keywords:
Dhillon-Gompertz Distribution,
Failure Rate,
Additive Methodology,
Reliability Analysis,
Moment,
Maximum Likelihood,
Monte Carlo Simulations

INTRODUCTION

Classical probability distribution has long been employed to draw inferences about populations based on observed data. However, it is widely recognized that some of these distributions fail to adequately capture the underlying patterns present in real-world datasets. The Gompertz distribution is one of the most important distributions with a wide range of applications in statistical practice. It is characterized by a shape that increases rapidly at first and then slows down. It's commonly used in survival analysis, particularly in modelling human mortality rates, but it has applications in various other fields such as reliability engineering and demography. The Gompertz distribution has being considered as a well-known model with an increasing failure rate function that investigated the modelling a variety of data. It is monotonically increasing or decreasing, depending on its parameter and this makes it less flexible compared to other generalized or modified models that can model bathtub-shaped or non-monotonic failure rate.

The cumulative distribution function and its corresponding probability density function of the Gompertz distribution is given by

$$F(x) = 1 - e^{\frac{-\alpha}{\beta}(e^{\beta t}-1)} \tag{1}$$

$$f(x) = \alpha e^{\beta t} e^{\frac{-\alpha}{\beta}(e^{\beta t}-1)} \tag{2}$$

For t > 0, $\alpha, \beta > 0$.
Where α and β are the scale and shape parameters respectively.

Numerous modifications of the Gompertz distribution have been carried out by researchers. El-Gohary et al. 2013) introduced a generalized Gompertz distribution with three parameters. Furthermore, Sarhan et al. (2016) developed the Exponentiated modified Weibull extension distribution, which generalizes the model introduced by El-Gohary et al. (2013).

How to cite this article: Usman U., Shamsuddeen S. & Faisal M. A. (2026). A New Four Parameter Hybrid of Dhillon-Gompertz lifetime distribution with Bathtub-shaped Failure Rate: Properties and Applications. *Journal of Basics and Applied Sciences Research*, *4*(1), 18-30. https://dx.doi.org/10.4314/jobasr.v4i1.3

Additionally, Ali et al. (2019), proposed a four-parameter extension known as the beta-Gompertz distribution, which encompasses several well-known lifetime distributions such as the beta-exponential and generalized Gompertz distributions as special cases. Adeyemi and Adeleke (2021), introduced a new probability distribution named Gompertz exponential pareto distribution with the properties and applications to bladder cancer and hydrological datasets using the Gompertz generator. Abba and Wang (2023) introduced a new flexible additive model that adequately describes complex reliability and survival data. It is called the Flexible Exponential Power-Gompertz (FEPG4) distribution. Jabril et al. (2024), proposed a new statistical model named the Generalized Gompertz-G family of distribution using a method introduced by Alzaatreh.

Several generalized models have been proposed to analyze lifetime data exhibiting non-monotone failure rates, particularly in reliability engineering Abba and Wang (2023). Examples of these models include the upper truncated Weibull distribution Zhang and Xie (2011), Chen's family of survival distributions by Chaubey and Zhang (2015), and the Beta Sarhan-Zaindin modified Weibull distribution by Saboor et al. (2016). Others are; A new weighted Gompertz distribution by Bakouch and Abd-El-Bar (2017), On some life distributions with flexible failure rate by Lu and Chiang (2018), A new extension of the exponential power distribution by Shakil et al. (2018), The log-normal modified Weibull distribution by Shakhatreh et al. (2019), A new extension of the topp–Leone-family of models by Muhammad et al. (2022), Classical and Bayesian estimations of improved Weibull–Weibull distribution by Wang et al. (2022), New Generalized odd Frechet-Odd Exponential-G Family of distribution by Sadiq et al. (2023) Chen-Burr XII Model as a Competing Risks Model by Kalantan et al. (2024). Despite their utility, research has revealed that many of these distributions display bathtub failure rate shapes but lack a relatively constant phase (Abba et al. 2022; Shakhatreh et al. 2019). This constant phase, which represents the useful life span of a component or system, is crucial for reliability modeling (Kosky et al. 2021). Therefore, constructing models that accurately capture this constant failure rate phase is essential. Merging the failure rates of two distributions is a powerful technique for creating more flexible and adaptable models that can capture complex patterns in data (Travirdizade and Ahmadpour 2021). This approach allows researchers to combine the strengths of different distributions, resulting in models that can handle a wide range of data behaviors. Building on this idea, [26] developed the Additive Weibull model by combining the hazard rates of two Weibull distributions.

On the other hand, Dhillon (1980), established a new distribution with hazard rate models which are expected to be useful in situations where the traditional distributions are rather cumbersome to apply or they failed to represent the failed failure data. The author symbolised the model as the FR model and described that the model may be utilized for describing various kinds of Monotonic Failure rate (MFR) and Non-Monotonic Failure Rate (NMFR). One of the weaknesses of the Dhillon distribution is that the Author neither derives the full log-likelihood function nor developed the score equations. Recently, modifications and extensions of the Dhillon were proposed by several researchers including a Modified Dhillon distribution by Iliyasu et al. (2025) which was developed using Beta integrated model approach, a four-parameter Bi-failure modes model by Abba et al. (2025a) in which the distribution of failure times due to one failure type follow Dhillon distribution, while the other failure type follows an exponential-power distribution. Abba et al (2025b) used the same methodology to construct a Weibull-Dhillon competing risk model. A robust multi-risk model and its reliability relevance by Abba et al. (2025c) with more robustness than a Weibull-Dhillon. The Additive Dhillon-Chen distribution by Amiru et al. (2025) which crossbred Dhillon and Chen distributions using additive methodology, the distribution can model both monotonic and non-monotonic failure rate pattern. Ibrahim and Aminu (2025) proposed a two-component Perks-Dhillon competing risk model by hybridizing Perks and Dhillon failure rate model. In this paper, we propose a new four-parameter lifetime distribution, called A New Four-parameter Hybrid Dhillon-Gompertz (HDG) distribution, which combines the failure rates of the Dhillon and Gompertz distributions in a serial system. The HDG distribution is designed to be highly flexible, allowing it to model data with both monotonic (consistent patterns) and non-monotonic (irregular patterns) behaviors.

MATERIALS AND METHODS

Proposed Hybrid Dhillon-Gompertz Distribution

Let us consider a system with two components arranged and functioning in a series, each component is operating independently at a given time t. The system fails when the first component fails. In view of this, our new model signifies the lifetime of the entire serial system with two components. The first component's lifetime follows a Dhillon distribution with parameters λ and θ, while the second component's lifetime follows a Gompertz distribution with parameters α and β. The complete system's lifetime is determined by the minimum lifetime of the two components. In other word, let T_1 represent the lifetime of the first component, which follows Dhillon distribution with parameter λ and θ, and let T_2 represent the lifetime of the second component which follows

Gompertz with parameter α and β. If the system lifetime is T, then $T = \min(T_1, T_2)$ has the cumulative distribution function given by

$$F(t) = 1 - e^{-\ln(\lambda t^{\theta}+1) - \frac{\alpha}{\beta}(e^{\beta t}-1)}, \qquad (3)$$

and

$$f(t) = \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} + \alpha e^{\beta t}\right) e^{-\ln(\lambda t^{\theta}+1) - \frac{\alpha}{\beta}(e^{\beta t}-1)} \qquad (4)$$

For $t, \lambda, \theta, \alpha > 0$ and $\beta \geq 0$.
Where $\lambda > 0$, $\alpha > 0$ are the scale parameters and $\theta, \beta \geq 0$ are the shape parameters respectively.

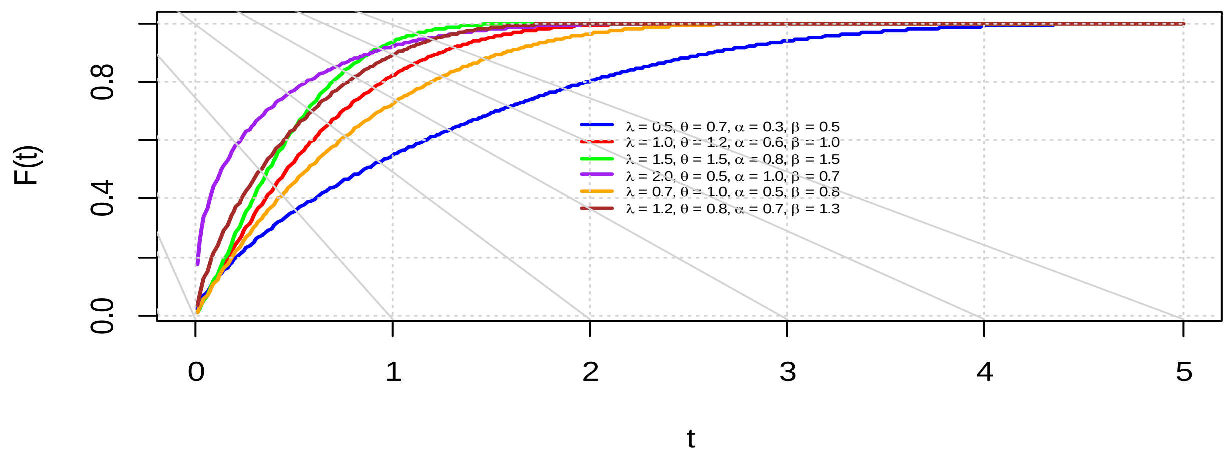

Figure 1. Plot of the HDG cumulative distribution function for some values of parameter.

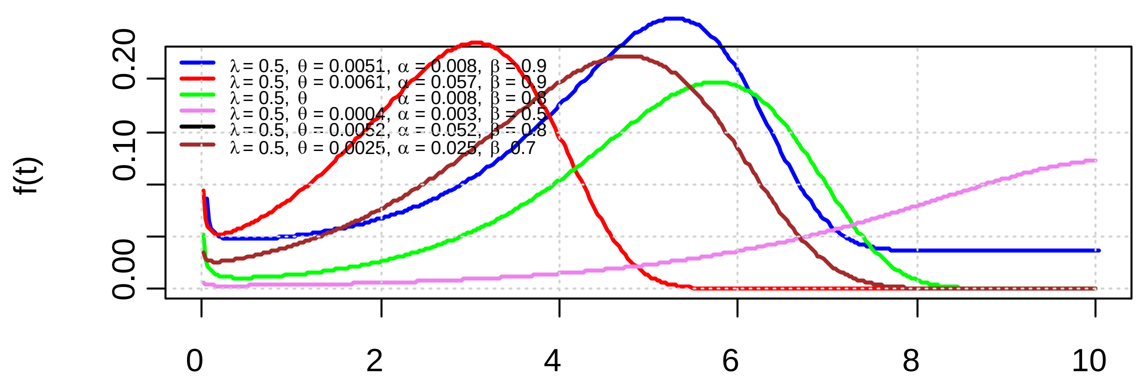

Figure: 2. Plot of the HDG probability density function for some values of parameter.

The FR function as well as the survival function of the proposed HDG distribution is defined as

$$h(t) = \frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} + \alpha e^{\beta t} \qquad (5)$$

$$S(t) = e^{-\ln(\lambda t^{\theta}+1) - \frac{\alpha}{\beta}(e^{\beta t}-1)} \qquad (6)$$

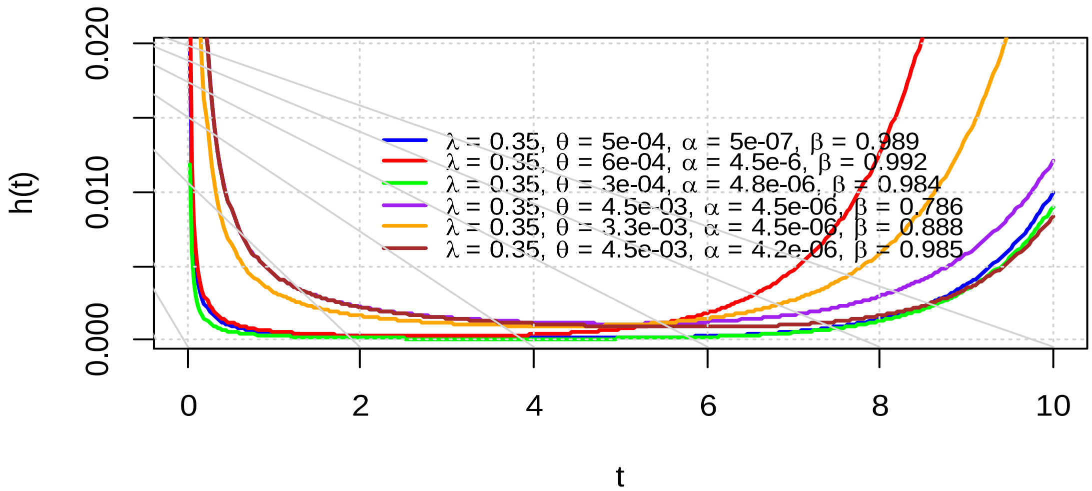

Figure: 3. Plot of the HDG failure rate function for some values of parameter.

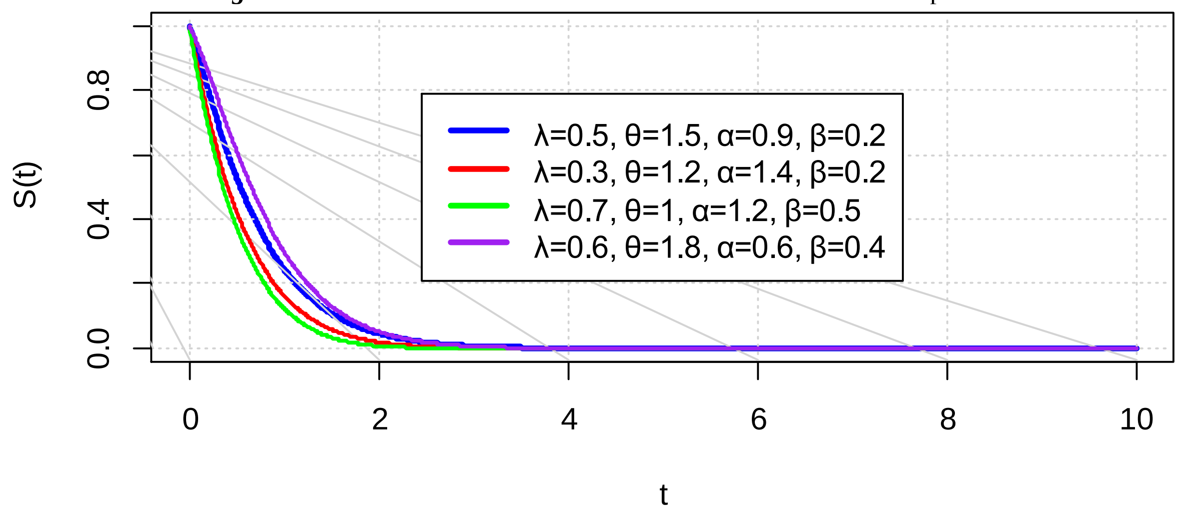

Figure: 4. Plot of the HDG Survival function for some values of parameters.

Measure of central Tendency

In this section, we present the measures of central tendency for the HDG distribution.

Quartile Function

Suppose the random variable T has the HDG distribution, then the quantile function T is given by $Q(p) = F^{-1}(p)$, which implies that finding t such that $F(t) = p$, for $p \in [0,1]$. That is,
Suppose that $F(t) = p$, Then

$$F(t) = 1 - e^{-ln(\lambda t^{\theta}+1) - \frac{\alpha}{\beta}(e^{\beta t}-1)} = p \quad (7)$$

This implies that

$$e^{-ln(\lambda t^{\theta}+1) - \frac{\alpha}{\beta}(e^{\beta t}-1)} = 1 - p \quad (8)$$

Taking the natural log of both sides, we have

$$-ln(\lambda t^{\theta}+1) - \frac{\alpha}{\beta}(e^{\beta t}-1) = \ln(1-p) \quad (9)$$

Simplify equation (20) by eliminating the negative sign, lead to

$$ln(\lambda t^{\theta}+1) - \frac{\alpha}{\beta}(e^{\beta t}-1) = -\ln(1-p) \quad (10)$$

$$ln(\lambda t^{\theta}+1) = -\ln(1-p) + \frac{\alpha}{\beta}(e^{\beta t}-1). \quad (11)$$

The above equation does not have a closed-form solution due to its complex dependence on t in both $ln(\lambda t^{\theta}+1)$ and $e^{\beta t}$. Therefore, a numerical method such as Newton Raphson method, bisection method, for finding the $Q(P)$ for specific parameter values and probability is appropriate.

Mode

The mode of the HDG distribution is obtained by differentiating the density function in equation (4) with respect to t.

$$f(t) = \int_0^\infty \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}, \quad (12)$$

$$f(t) = g(t).h(t), \quad (13)$$

where

$$g(t) = \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) \ and$$

$$h(t) = e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}. \quad (14)$$

The mode of HDG distribution is given by

$$f'(t) = h(t)\left(\frac{(\lambda t^\theta+1)(\lambda\theta(\theta-1)t^{\theta-2}) - \lambda^2\theta^2 t^{2\theta-2})}{(\lambda t^\theta+1)^2} + \alpha\beta e^{\beta t}\right) + g(t)\left(\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}\right) \quad (20)$$

Asymptotic Behavior of HDG Distribution

In this section, we investigate the behavior of the HDG model. The limit of equation (4) as $t \to 0$ and as $t \to \infty$ i.e.

$$\lim_{t\to 0} f(t) = \lim_{t\to\infty} f(t) = 0 \quad (21)$$

$$f(t) = \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) \text{X}\ e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)} \quad (22)$$

This implies that

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) \text{X}\ e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}\right] \quad (23)$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) \text{X} \lim_{t\to 0} e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)} \quad (24)$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) \text{X}\ 0 \quad (25)$$

$$\lim_{t\to 0} f(t) = 0 \quad (26)$$

Also,

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) \text{X}\ e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}\right] \quad (28)$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) \text{X} \lim_{t\to\infty} e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)} \quad (29)$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) \text{X}\ 0 \quad (30)$$

$$\lim_{t\to\infty} f(t) = 0 \quad (31)$$

Therefore, the HDG distribution has a unimodal.

Order Statistics

Let $T_1, T_2, \ldots, T_n$ be a random sample from the HDG distribution and $T_{k:n}$ is the k^{th} order statistic of the sample, then the PDF of $T_{k:n}$ is given by

$$f_{k:n}(t) = \frac{1}{B(k,n-k+1)}[F(t)]^{k-1}[1 - F(t)]^{n-k} f(t) \quad (32)$$

$$where, \quad B(k,n-k+1) = \frac{\Gamma(k)\Gamma(n-k+1)}{\Gamma(n-k+1+k)} = \frac{\Gamma(k)\Gamma(n-k+1)}{\Gamma(n+1)} = \frac{(k-1)!\,(n-k)!}{n!}$$

Therefore, equation (32) can be re-written as

$$f_{k:n}(t) = \frac{1}{\left\{\frac{(k-1)!\,(n-k)!}{n!}\right\}}[F(t)]^{k-1}[1 - F(t)]^{n-k} f(t) = \frac{n!}{(n-k)!(n-k)!}\ [F(t)]^{k-1}[1-F(t)]^{n-k} f(t) \quad (33)$$

Where $F(t)$ and $f(t)$ are given in equation (3) and (4) respectively.

$$[F(t)]^{k-1} = \left[1 - e^{-\left\{ln(\lambda t^\theta+1)+\frac{\alpha}{\beta}(e^{\beta t}-1)\right\}}\right]^{k-1} \quad (34)$$

$$= \sum_{i=0}^{k-1}(-1)^i \binom{k-1}{i} e^{-i\left\{ln(\lambda t^\theta+1)+\frac{\alpha}{\beta}(e^{\beta t}-1)\right\}} \quad (35)$$

Similarly,

$$[1-F(t)]^{n-k} = \left[e^{-\left\{ln(\lambda t^\theta+1)+\frac{\alpha}{\beta}(e^{\beta t}-1)\right\}}\right]^{n-k} = e^{-(n-k)\left\{ln(\lambda t^\theta+1)+\frac{\alpha}{\beta}(e^{\beta t}-1)\right\}} \quad (36)$$

Inserting (33), (35) and (36) in (32), the pdf of the k^{th} order statistics can be given as in (37)

$$f_{k:n}(t) = \left(\begin{array}{c} \frac{n!}{(n-k)!(n-k)!}\sum_{i=0}^{k-1}(-1)^i \binom{k-1}{i} e^{-i\left\{ln(\lambda t^\theta+1)+\frac{\alpha}{\beta}(e^{\beta t}-1)\right\}} \\ \times\left\{e^{-(n-k)\left\{ln(\lambda t^\theta+1)+\frac{\alpha}{\beta}(e^{\beta t}-1)\right\}}\right\} \\ \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) e^{-\left\{ln(\lambda t^\theta+1)+\frac{\alpha}{\beta}(e^{\beta t}-1)\right\}} \end{array}\right) \quad (37)$$

Characteristics Function of HDG Distribution

Let T be a random variable that follows HDG distribution with density function given in equation (4). The characteristic function of T is defined as

$$\varphi_T(u) = E(e^{iut}) = \int_{-\infty}^{\infty} e^{iut} f(t)dt \quad (38)$$

Substitute (4) into equation (38)

$$= \int_{-\infty}^{\infty} e^{iut}\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha e^{\beta t}\right) e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)} dt \quad (39)$$

$$= \int_{-\infty}^{\infty} e^{iut}\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1}\right) e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)} dt + \int_{-\infty}^{\infty} e^{iut}\left(\alpha e^{\beta t}\right) e^{-ln(\lambda t^\theta+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)} dt \quad (40)$$

Recall that, $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$

By applying series expansion and letting $x = \lambda t^{\theta}$ on $\frac{1}{\lambda t^{\theta}+1}$, we have

$$\frac{1}{\lambda t^{\theta}+1} = \sum_{c=0}^{\infty}(-1)^c (\lambda t^{\theta})^c \quad (41)$$

$$E(e^{iut}) = \int_{-\infty}^{\infty} e^{iut}\lambda\theta t^{\theta-1}\sum_{c=0}^{\infty}(-1)^c\lambda^c t^{\theta c} e^{-ln(\lambda t^{\theta}+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}dt + \int_{-\infty}^{\infty} e^{iut}\alpha e^{\beta t}\sum_{c=0}^{\infty}(-1)^c\lambda^c t^{\theta c} e^{-\frac{\alpha}{\beta}(e^{\beta t}-1)}dt \quad (42)$$

$$\lambda\theta\sum_{c=0}^{\infty}(-1)^c\lambda^c\int_{-\infty}^{\infty} e^{iut} t^{\theta-1+\theta c} e^{-ln(\lambda t^{\theta}+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}dt + \sum_{c=0}^{\infty}(-1)^c\lambda^c t^{\theta c}\int_{-\infty}^{\infty} e^{iut}\alpha e^{\beta t} e^{-\frac{\alpha}{\beta}(e^{\beta t}-1)}dt \quad (43)$$

Hence, the characteristics function of HDG distribution is in equation (43).

Moment and Moment Generating Function of HDG Distribution

The moment generating function of T, $M_T(x) = E(e^{tx})$ is given by

$$M_T(x) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(t)dt \quad (44)$$

Applying Taylor series to expand e^{tx} and have

$$e^{tx} = \sum_{i=0}^{\infty}\frac{x^i}{i!}t^i \quad (45)$$

Therefore,

$$M_T(x) = \int_0^{\infty}\sum_{i=0}^{\infty}\frac{x^i}{i!}t^i f(t)dt \quad (46)$$

$$= \sum_{i=0}^{\infty}\frac{x^i}{i!}\int_0^{\infty} t^i f(t)dt = 1 + \sum_{i=0}^{\infty}\frac{x^i}{i!}E(t^i)dt \quad (47)$$

To find the i^{th} moment about the origin i.e. u_i',

$$u_i' = E(t^i) = \int_0^{\infty} t^i f(t)dt \quad (48)$$

$$= \int_0^{\infty} t^i\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} + \alpha e^{\beta t}\right)e^{-ln(\lambda t^{\theta}+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}dt \quad (49)$$

Recall that $\frac{1}{\lambda t^{\theta}+1}$, $\sum_{c=0}^{\infty}(-1)^c (\lambda t^{\theta})^c$,

therefore $\frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} = \lambda\theta t^{\theta-1}\sum_{c=0}^{\infty}(-1)^c\lambda^c$

This implies that

$$\frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} = \sum_{c=0}^{\infty}(-1)^c\lambda^{c+1}\theta\, t^{(c+1)\theta-1} \quad (50)$$

Substitute equation (50) into equation (49) and have

$$u_i' = \int_0^{\infty} t^i\left(\sum_{c=0}^{\infty}(-1)^c\lambda^{c+1}\theta\, t^{(c+1)\theta-1} + \alpha e^{\beta t}\right)e^{-ln(\lambda t^{\theta}+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}dt \quad (51)$$

$$= \sum_{c=0}^{\infty}(-1)^c\lambda^{c+1}\theta\int_0^{\infty} t^{i+(c+1)\theta-1}e^{-ln(\lambda t^{\theta}+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}dt + \alpha\int_0^{\infty} t^i e^{\beta t}e^{-ln(\lambda t^{\theta}+1)-\frac{\alpha}{\beta}(e^{\beta t}-1)}dt \quad (52)$$

Now, we let $m = \frac{\alpha}{\beta}e^{\beta t}$ to simplify further and evaluate u_i' by transforming the integral using substitution method

$m = \frac{\alpha}{\beta}e^{\beta t}$, *implies that* $t = \frac{1}{\beta}\ln\left(\frac{m\beta}{\alpha}\right)$, and $dt = \frac{1}{\beta}dm$

$$u_i' = \sum_{c=0}^{\infty}(-1)^c\lambda^{c+1}\theta\,\nabla_1(m) + \alpha\nabla_2(m) \quad (53)$$

Where $\nabla_1(m)$ *and* $\alpha\nabla_2(m)$ are the integrals over m.

Now, the MGF is denoted by

$$M_T(x) = 1 + \sum_{c=0}^{\infty}\frac{x^i}{i!}(-1)^c\lambda^{c+1}\theta\,\nabla_1(m) + \alpha\nabla_2(m) \quad (54)$$

Parameter Estimation

In this section, we employ the maximum likelihood estimation (MLE) method to estimate the parameters of the HDG distribution.

Maximum Likelihood Estimator

Suppose that the likelihood function for a dataset of independent observations $t_1, t_2, \dots\dots, t_n$ form the density function $f(t)$ is given by

$$Lf(\omega) = \prod_{i=1}^{n} f(t_i) \quad (55)$$

$$Lf(\omega) = \prod_{i=1}^{n}\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} + \alpha\beta t^{\beta-1}e^{t^{\beta}}\right)e^{-ln(\lambda t^{\theta}+1)+\alpha(1-e^{t^{\beta}})} \quad (56)$$

Taking the natural log of the likelihood function and simplify it by turning products into sums, we have

$$\ell f(\omega) = \sum_{i=1}^{n} ln\ f(t_i) \quad (57)$$

$$\ell f(\omega) = \sum_{i=1}^{n}\left[\ln\left(\frac{\lambda\theta t_i^{\theta-1}}{\lambda t_i^{\theta}+1} + \alpha\beta t_i^{\beta-1}e^{t_i^{\beta}}\right) - ln(\lambda t_i^{\theta}+1) + \alpha(1-e^{t_i^{\beta}}\right] \quad (59)$$

We can derive the score equations by taking the partial derivatives of (59) with respect to λ, θ, α, and β and equate to zero, which eventually produce MLEs ($\hat{\lambda}, \hat{\theta}, \hat{\alpha}, \hat{\beta}$) of ($\lambda$, θ, α, β). Yet, the score equations from (59) may not lead to closed-form solutions, and hence in this research, we recommend using any of the statistical software available like R package to maximize the log-likelihood function in (59)

RESULTS AND DISCUSSION

In this session, we present the empirical evaluation of the proposed HDG model using well-known datasets representing failure times and survival times data. The objective is to assess the flexibility and reliability of the proposed model in capturing data patterns characterized by bathtub-shaped failure rates. Each dataset is analyzed and compared with other existing models previously discussed in the literature. The comparison is based on model selection criteria, including Log-likelihood, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Corrected Akaike Information Criterion (AICc). The model with the smallest values across these criteria is considered the best-fitting model. In addition to numerical comparison, graphical tools such as histograms and Total Time on Test

(TTT) plots are used to examine the shape of the failure rate and validate model suitability.

Modalities of the Simulation Studies

In this part, we perform a Monte Carlo simulation to evaluate the performance of maximum likelihood estimation (MLE) for the proposed HDG distribution. First, the probability density function (PDF) of the distribution is specified, from which the cumulative distribution function (CDF) is obtained numerically by integration. An inverse CDF (quantile function) is then constructed using interpolation, allowing random sample generation from the distribution. Samples are generated for different sample sizes (30, 60, 90, and 120) using fixed true parameter values ($\lambda = 0.2$, $\theta = 2$, $\alpha = 0.2$, $\beta = 1.0$), and each experiment is replicated 100 times for statistical stability. For each simulated dataset, the parameters are estimated through MLE by minimizing the negative log-likelihood function using the nlminb optimization routine under specified bounds. The estimated values are compared with assumed true parameters to compute the bias and mean squared error (MSE) for each parameter across replications. The results are summarized in three matrices; Esti (mean estimates), Biass (biases), and Meansq (MSEs), which show how estimation accuracy and precision improve as the sample size increases. This approach provides a comprehensive numerical assessment of the estimators' performance for the proposed model.

Table 1 Estimators' performance based on Average mean, Bias and MSE of the HDE parameters at four distinct sample size.

Sample Size	$\lambda = 0.2$	$\theta = 1.8$	$\alpha = 0.3$	$\beta = 0.15$	$\lambda = 0.2$	$\theta = 1.8$	$\alpha = 0.3$	$\beta = 0.15$	$\lambda = 0.2$	$\theta = 1.8$	$\alpha = 0.3$	$\beta = 0.15$
	Estimated Means				Bias				MSE			
30	0.2105	1.9696	0.1871	1.0743	0.0105	0.1696	-0.1129	0.9243	0.0020	0.2119	0.0143	0.8733
60	0.2173	1.9711	0.1942	1.0433	0.0173	0.1712	-0.1058	0.8933	0.0019	0.1950	0.0125	0.8733
90	0.2157	1.9885	0.1921	1.0239	0.0157	0.1775	-1079	0.8739	0.0020	0.1955	0.0127	0.7732
120	0.2104	1.9265	0.1902	1.0359	0.0104	0.1265	-0.1098	0.8859	0.0019	0.1625	0.0131	0.7931

Table 1 presents the results of a Monte Carlo simulation conducted to assess the performance of the Maximum Likelihood Estimators (MLEs) for the parameters $\lambda = 0.2$, $\theta = 1.8$, α=0.3, and $\beta = 0.15$ of the proposed distribution under varying sample sizes (n = 30, 60, 90, 120). The estimated means, biases, and mean squared errors (MSEs) are reported for each parameter. As the sample size increases, the estimated means of all parameters move closer to their true values, while both the bias and MSE values generally decrease, indicating improved estimation accuracy and consistency. For smaller samples, such as n = 30, larger biases and MSEs are observed, particularly for θ and α, suggesting greater variability and slight overestimation. However, with larger samples (n = 90, 120), the estimates stabilize, and the biases become minimal, confirming that the MLEs are asymptotically unbiased and efficient. Furthermore, the results demonstrate that the Maximum Likelihood Estimation method performs well for the HDG model, providing reliable parameter estimates that improve with increasing sample size.

Applications

In this section, we utilized some existing real-life data sets to evaluate the flexibility of our proposed HDG distribution.

Early Cable-Joint Failure Data

Table 2 presents 16 early cable-joint failure (ECJF) times. The data was reported by Tang *et al.* (2015) and was utilized by several authors as a benchmark for testing lifetime distributions with bathtub-shaped or non-monotonic failure rate (FR) functions, owing to its clear reflection of early failure behaviour. Figure 4 shows that the data exhibits a bathtub-shaped failure rate, confirming that models allowing for non-monotonic FR functions (e.g., HDE, OCE, hybrid models) are more appropriate than the simple exponential distribution.

Table 2 Early cable-joint failure times data

5	43	65	194	259	262	354	620	968	2100	2629	2676	2676	2744	4254	4254

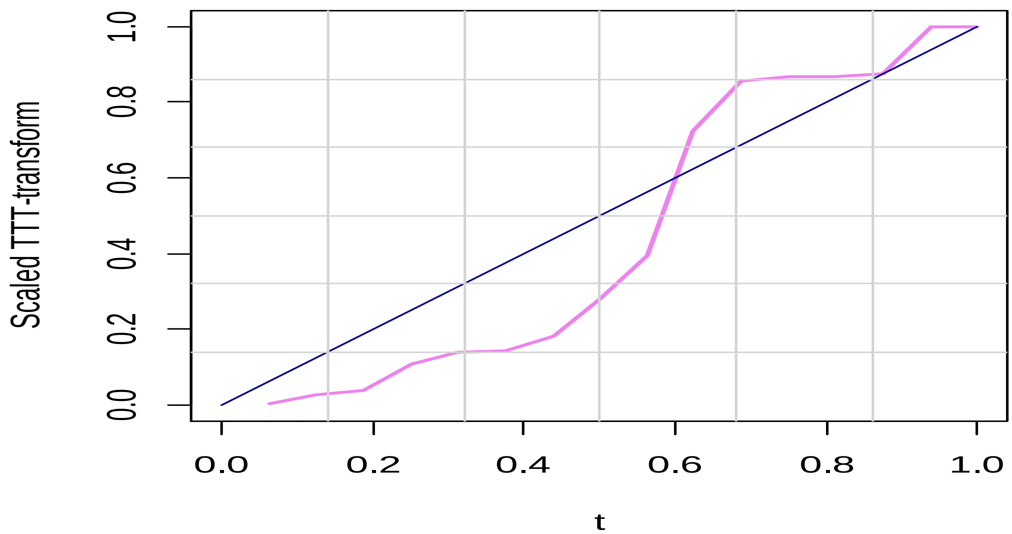

***Figure** 5* Empirical Scaled TTT-transform plot for ECJF data.

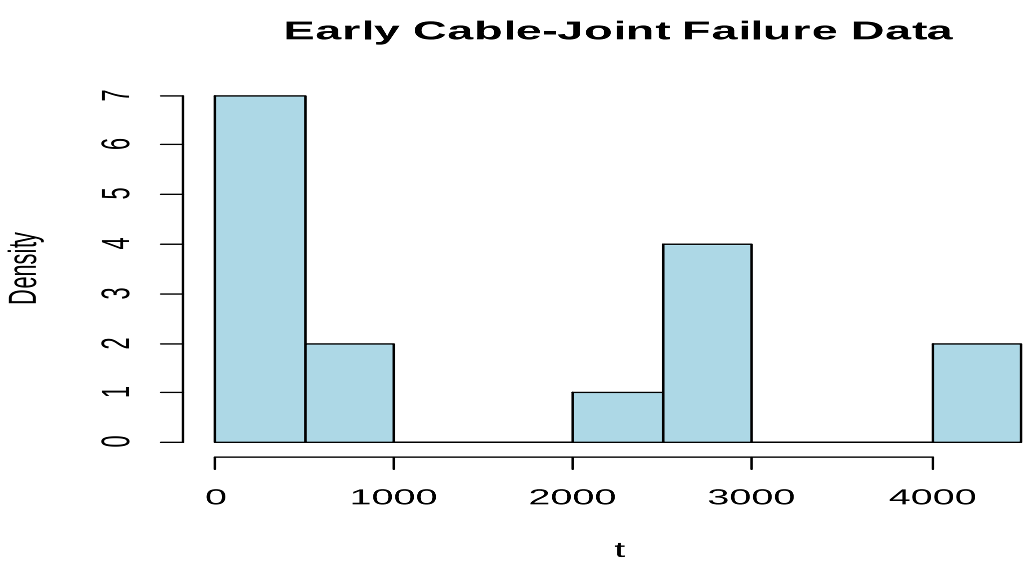

***Figure** 6* Histogram of ECJF data

Figure 6 shows the histogram of early cable-joint failure data with a high concentration of failures at the beginning (0–500), reflecting infant mortality due to initial defects, followed by a period of reliability with few failures, and then a resurgence of failures around 2000–3000 and beyond 4000, indicating wear-out.

Table 3 Descriptive statistics for ECJF data.

N	Min.	1st Qu.	Median	Mean	Sd.	3rd Qu.	Max.
16	5.0	242.8	794.0	1506.4	1520.59	2676.0	4254.0

Table 3 reveals that the dataset consists of 16 observations with failure times ranging from 5.0 to 4254.0. The first quartile (242.8) and median (794.0) indicate that half of the failures occur relatively early, while the mean (1506.4) is much larger, reflecting the influence of later failures. The third quartile (2676.0) and maximum (4254.0) show that some joints last much longer, creating a wide spread in the data, as confirmed by the large standard deviation (1520.6). Overall, the data suggest a mix of early and late failures with high variability.

Table 4 Parameters estimation of the HDG distribution and the five other existing distributions for the infected cases data.

Model	Parameters				
HDG	$\hat{\lambda}$ = 0.0075	$\hat{\theta}$ = 0.9464	$\hat{\alpha}$ = 0.0002	$\hat{\beta}$ = 0.0132	
AGW	$\hat{\alpha}$ = 0.6492	$\hat{\gamma}$ = 0.0006	$\hat{\lambda}$ = 0.1669	$\hat{\theta}$ = 709.20	
FACG	$\hat{\gamma}$ = 0.1781	$\hat{\alpha}$ = 0.0241	$\hat{\theta}$ = 0.705.30	$\hat{\lambda}$ = 0.9464	
AMW	$\hat{\gamma}$ = 7.7e-5	$\hat{\alpha}$ = 0.0140	$\hat{\lambda}$ = 0.0351	$\hat{\theta}$ = 0.5611	$\hat{\beta}$ = 149.30
INMW	$\hat{\alpha}$ = 0.0005	$\hat{\beta}$ = 8.6^{-5}	$\hat{\gamma}$ = 0.5925	$\hat{\theta}$ = 0.5.387	$\hat{\lambda}$ = 0.0118

Table 5 Goodness-of-fit test results of the HDG distribution and the four other existing distributions for the ECJF dataset.

MLE, -LL, AIC, AICc and BIC for the trained models on the ECJF dataset				
Model	-L	AIC	AICc	BIC
HDG	108.03	224.06	227.14	227.62
AGW	121.59	251.18	254.82	254.27
FACG	122.04	252.09	255.72	254.27
AMW	124.52	259.05	265.05	262.91
INMW	129.09	268.18	274.18	272.04

Table 5 above shows the MLEs of the models' parameters, along with the log-likelihood, AIC, AICc, and BIC statistics. It is observed that the HDG model has the smallest values for AIC, AICc, and BIC among the competing models, indicating that the HDG distribution provides the best fit for the dataset.

Survival times data for bladder cancer among 128 individuals

Table 6 presents the remission periods in months for bladder cancer among 128 individuals and used by many researchers including Marshal-Olkin (1997), Karakas and Bulut (2019) among others. Figure 7 shows the Total Time on Test (TTT) plot, which is used to visualize the nature of the failure rate (hazard function) in reliability analysis. The diagonal line represents a constant failure rate, while the purple curve represents the empirical TTT transform based on the observed data. Since the curve lies slightly below the diagonal in the early part and above it in the latter part, it suggests a decreasing failure rate at the beginning followed by an increasing failure rate later on, which is characteristic of a bathtub-shaped FR function.

Table 6 Survival times data for bladder cancer among 128 individuals.

0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4
1.46	1.76	2.02	2.02	2.07	2.09	2.23	2.26	2.46	2.54	2.62	2.64
2.69	2.69	2.75	2.83	2.87	3.02	3.25	3.31	3.36	3.36	3.48	3.52
3.57	3.64	3.7	3.82	3.88	4.18	4.23	4.26	4.33	4.34	4.4	4.5
4.51	4.87	4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41	5.41	5.49
5.62	5.71	5.85	6.25	6.54	6.76	6.93	6.94	6.97	7.09	7.26	7.28
7.32	7.39	7.59	7.62	7.63	7.66	7.87	7.93	8.26	8.37	8.53	8.65
8.66	9.02	9.22	9.47	9.74	10.06	10.34	10.66	10.75	11.25	11.64	11.79
11.98	12.02	12.03	12.07	12.63	13.11	13.29	13.8	14.24	14.76	14.77	14.83
15.96	16.62	17.12	17.14	17.36	18.1	19.13	20.28	21.73	22.69	23.63	25.74
25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05				

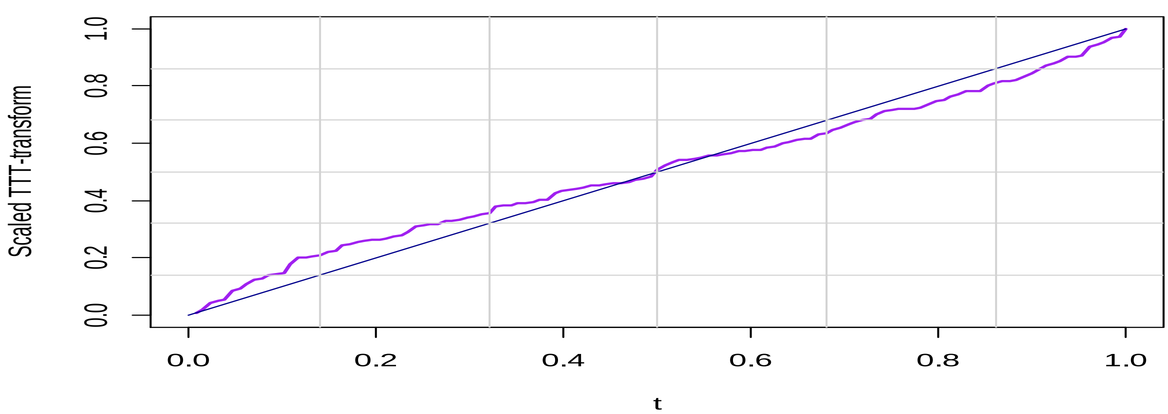

***Figure** 7* Empirical Scaled TTT-transform plot for bladder cancer among 128 individuals.

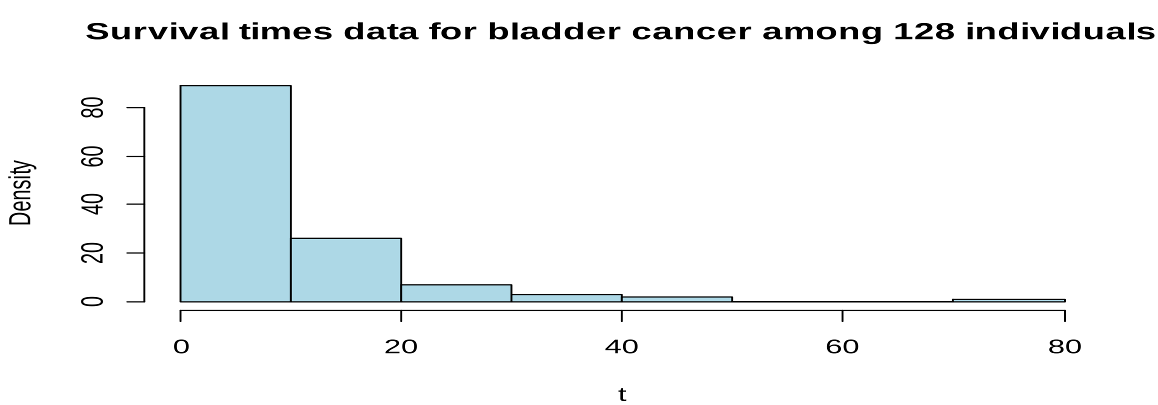

***Figure** 8* Histogram data for bladder cancer among 128 individuals.

Figure 8 shows the survival times of 128 individuals with bladder cancer, where the x-axis represents time (t) and the y-axis represents density. The distribution is highly right-skewed, with the majority of individuals experiencing relatively short survival times clustered between 0 and 20 units, and very few surviving beyond 40 units. This indicates that most patients had early failures or short survival durations, while a small fraction experienced much longer survival, creating a long tail in the distribution. Such a pattern suggests the presence of early high risk of death followed by a decreasing FR over time.

Table 7 Descriptive statistics for the 128 bladder cancer survival times

N	Min.	1st Qu.	Median	Mean	Sd.	3rd Qu.	Max.
128	0.080	3.348	6.395	9.66	10.51	11.838	79.050

Table 7 presents the summary statistics of the 128 bladder cancer survival times showing a wide variation in survival durations. The minimum survival time is very short (0.08), while the maximum extends to 79.05, indicating a long right tail. The first quartile (3.35) and median (6.40) suggest that at least half of the patients survived less than 7 months, indicating a concentration of early death. The mean survival time (9.66) is higher than the median, reflecting the skewness caused by a few individuals with much longer survival. The standard deviation (10.51) is relatively large compared to the mean, emphasizing high variability in the data. Overall, most individuals had short survival durations, but a small subset survived significantly longer, consistent with the skewed histogram pattern.

Table 8 Parameters estimation of the HDG distribution and the three other existing distributions for the infected cases data.

Model	Parameters			
HDG	$\hat{\lambda}$ = 0.0305	$\hat{\theta}$ = 1.7014	$\hat{\alpha}$ = 0.0251	$\hat{\beta}$ = 0.0008
MOPG	$\hat{a}$ = 0.1068	$\hat{b}$ = 0.000001	$\hat{c}$ = 1.5194	$\hat{d}$ = 0.1098
MOG	$\hat{a}$ = 0.1068	$\hat{b}$ = 0.000001	-	$\hat{d}$ = 0.9999
PG	$\hat{a}$ = 0.0939	$\hat{b}$ = 0.000001	$\hat{c}$ = 1.0478	-

Table 9 Goodness-of-fit test results of the HDG distribution and the three other existing distributions for the for bladder cancer among 128 individuals.

Model	L	AIC	AICc	BIC
HDG	818.7800	826.7800	827.1050	835.2088
MOPG	820.5461	828.5461	828,8713	839.9542
MOG	828.6841	834.6841	834.8792	843.2402
PG	1084.1740	10901740	1090.3691	1098.7300

Table 9 presents four fitted models (HDG, MOPG, MOG, and PG) with their respective information criteria values (AIC, AICc, and BIC). Lower values of these statistics indicate a better fit, accounting for model complexity. Among the models, HDG distribution yield the smallest AIC (818.78), AICc (826.78), and BIC (835.21), confirming that it provides the best fit to the bladder cancer survival data

CONCLUSION

In this article, we presented the empirical evaluation newly proposed HDG lifetime distribution developed using an additive methodology. The study assessed the flexibility and performance of the model using real-life datasets representing bathtub-shaped failure rate pattern (i.e. Decreasing, Constant and Increasing). The parameters of the proposed model were estimated using the Maximum Likelihood Estimation (MLE) method, and its goodness-of-fit was compared with that of existing lifetime models using the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Corrected Akaike Information Criterion (AICc). The empirical results revealed that the proposed HDG model outperformed the existing distributions across all datasets considered. For the Early Cable-Joint Failure (ECJF) dataset, the HDG distribution provided the best fit, outperforming the AGW**,** FACG**,** AMW, and INMW models, confirming its strength in handling engineering data with early-life failures. For the Bladder Cancer dataset, which exhibited a non-monotonic failure rate, the HDG distribution again achieved the best fit among competing models such as MOPG**,** MOG**,** and PG**,** thereby demonstrating its suitability for biomedical survival data characterized by early mortality and long-term survivorship. Furthermore, the findings indicate that the HDG distribution is highly flexible and reliable model capable of accurately describing a broad range of lifetime behaviors. Its superior goodness-of-fit across the two datasets underscores its robustness and practical utility in reliability engineering, industrial maintenance, and biomedical survival analysis.

Future research can investigate additional properties of the proposed HDG model such as Mean Residual Lifetime (MRL) to enhance understanding of the models' behavior. Alternative estimation techniques such as Bayesian, least squares, or percentile estimation may yield complementary insights.

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