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A Nonlinear Dynamical Systems Approach to Modelling Chaotic Flow Patterns in Reused Engine Oil Under Variable Shear and Thermal Stress

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ABSTRACT
Recycling of engine oil is a widespread practice in Nigeria, largely driven by high costs of fresh lubricants and limited availability. While reuse reduces waste and provides short-term economic relief, the process progressively leads to viscosity loss, thermal degradation, and the accumulation of contaminants such as soot, metallic particles, and oxidation byproducts. These changes destabilize flow characteristics and compromise lubrication efficiency. To investigate this phenomenon, a mathematical framework based on nonlinear dynamical systems was developed. The model integrates the incompressible Navier–Stokes equations with a viscosity degradation law, contaminant transport, and stochastic forcing to reflect random disturbances from impurities. Thermal effects were incorporated through an energy equation to simulate harsh operating conditions typical of tropical climates. Numerical simulations were carried out using a finite-difference scheme and Euler–Maruyama integration for stochastic terms, implemented in the Dedalus framework. Results showed that reused oil exhibits a transition from laminar flow to chaotic regimes through bifurcations, particularly under high shear stress and elevated temperatures. Lyapunov exponent analysis confirmed the onset of chaos, while phase portraits revealed strange attractors characteristic of complex dynamics. An optimization routine estimated safe reuse limits, predicting maximum operation cycles of ~27 hours before instability occurs. This study highlights the critical influence of contaminants and thermal stress in triggering chaotic flow, providing theoretical guidance for safer and more sustainable lubricant management in resource-constrained environments.

Keywords:
Reused engine oil;
Chaotic flow behavior;
Lubrication performance;
Navier-Stokes equations;
Nonlinear dynamics

INTRODUCTION

Engine oil reuse is a common practice motivated by financial constraint, restricted availability of new lubricants, and insufficient waste management facilities (Okeniyi et al., 2015). The automotive sector in Nigeria is a major economic driver that relies majorly on commercial vehicles, power generators, and small-scale mechanics, all of which frequently employ reused oil to reduce operational costs (Afolabi et al., 2018). Although this method reduces waste and promotes sustainability, it presents serious problems because of changes in the qualities of the oil, such as decreased viscosity, thermal degradation, and the accumulation of impurities including soot, metal particles, and byproducts of oxidation (Bhushan, 2013).

These changes can affect lubrication performance, increasing wear, friction, and the risk of engine failure, especially in Nigeria's difficult operating conditions with high temperatures and high engine loads (Stachowiak & Batchelor, 2013).

According to Adebayo and Iweala (2019), one of the unethical practices in Nigeria's local automotive industry is the recycling of oil using conventional filtration techniques, including cloth or simple sieves, which lack standardized quality control. This presents significant questions regarding engine longevity and efficiency, particularly in high-demand applications like generators and commercial buses that are vital to Nigeria's economy. Although earlier research has looked at used oil by measuring its viscosity, chemical makeup, or wear patterns empirically, these studies usually utilize linear models that are unable to account for the intricate, transient flow dynamics under high-shear and thermal conditions (Rudnick, 2017).

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Degraded viscosity, heat cycling, and stochastic perturbations from contaminants can all work together to create nonlinear flow behaviors, such as chaotic regimes with highly sensitive and unpredictable dynamics (Pope, 2000). A major problem in Nigeria, where economic pressures encourage prolonged oil reuse, is that such chaotic flows may damage lubrication films, speeding up mechanical wear and shortening engine lifespan (Lahr, 2016; Strogatz, 2018, Xu et al., 2023.).

Even though reusing oil is common in Nigeria, mathematical modeling has not been used to thoroughly investigate the possibility of chaotic flow patterns in degraded oil. By creating a novel framework specifically suited to Nigeria, this study fills this gap by examining chaotic flow in used engine oil.

To take into consideration the contaminants that are common in used oil, a system of nonlinear PDEs based on the Navier-Stokes equations was created. This system included stochastic forcing and a viscosity degradation model. Using Lyapunov exponents and bifurcation analysis, this research aim to characterize the onset of chaotic flow regimes; quantify the influence of contaminants, thermal effects, and shear stress on flow stability; and create an optimization framework to maximize oil reuse cycles while avoiding chaotic flows that could affect engine performance. This study applies chaos theory to a real-world fluid dynamics issue and offers practical advice for sustainable oil management in Nigeria's industrial and automotive sectors, striking a balance between engine reliability and financial requirements (Bird et al.(1987), Babalola, & Iyorzor (2017)).

MATERIALS AND METHODS

2. Mathematical Model

2.1 Governing Equations

A thin film in a bearing with dimensions $L_x \times L_y = 10^{-2}\,m \times 10^{-3}$ m for two-dimensional (2D) lubrication flow of reused engine oil was modeled. A two-dimensional model of lubricant flow within a thin bearing film was developed to simulate the behavior of reused engine oil, drawing on foundational concepts in elasto-hydrodynamic lubrication as outlined by Dowson and Higginson (2014). The incompressible Navier-Stokes equations govern the velocity field u = (u, v) and pressure (p) (Ju, Yan, & Sun, 2022):

$$\rho\left(\frac{\partial u}{\partial t}+(u\cdot\nabla)u\right)=-\nabla p+\nabla\cdot\left(\mu\left(x,t,T\right)\nabla u\right)+f \qquad (1)$$

$$\nabla\cdot u=0$$

where ρ= 850 m^3 /kg is the density, μ(x, t, T) is the dynamic viscosity, and f is the external force. The viscosity is modeled (Sutcliffe, (2011)) :

$$\mu(x,t,T)=\mu_0\exp(-k_1t-k_2T+k_3C(x,t)) \qquad (2)$$

with initial viscosity μ_0 = 0.05 Pa·s, degradation rate k_1 = 0.01 s^{-1}, thermal sensitivity k_2 = 0.02 K^{-1}, and contaminant influence k_3 = 0.1 m^3/kg. The contaminant concentration C(x, t) evolves according to the advection-diffusion equation:

$$\frac{\partial C}{\partial t}+u\cdot\nabla C=D\nabla^2 C \qquad (3)$$

where

D = 10^{-6} m^2/s is the diffusion coefficient.To account for stochastic effects of contaminants (e.g., soot, metal particles), the external force includes a deterministic shear component and random noise:

$$f=f_d+\xi(x,t) \qquad (4)$$

where

$f_d=\gamma e_x$applies a shear force with rateγ, and$\xi(x,t)$is Gaussian white noise with correlation:

$$\xi(x,t)\xi(x',t')=\sigma 2\delta(x-x')\delta(t-t') \qquad (5)$$

and noise amplitude σ= 0.01 N/m^2.Thermal effects are modeled using the energy equation (Pope, 2000):

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\cdot\nabla T\right)=K\nabla^2 C+\mu\Phi \qquad (6)$$

where c_p = 2000 kg·K is the specific heat, k = 0.15 W/m·K is the thermal conductivity, and $\mu\Phi=(\nabla u:\nabla u)$is the viscous dissipation rate (Cellini, Peterson, & Porfiri, 2017, Chapron et al., 2024).

2.2 Nondimensionalization

To simplify analysis, nondimensionalize using characteristic scales was done using the following defined parameters: length L = 10^{-3}m, velocity U = 1m/s, and T_0 = 373K.

Reynolds number Re $=\frac{\rho UL}{\mu_0}$, the ratio of inertial to viscous forces (indicating flow stability Re = 17 for fresh oil, increasing with viscosity degradation).

Thermal parameter, $\alpha=K_2T_o$, quantifying viscosity's sensitivity to temperature, critical in Nigeria's hot climate (α = 7.46, scaled to 0.1–1 in simulations).

Stochastic intensity, $\beta=\frac{\sigma L}{\mu_o U}$, measuring contaminant-induced perturbations relative to viscous forces

(β = 0.0002, amplified to 0–0.1 in simulations).

3. Computational Methodology

3.1 Numerical Discretization

The PDEs are discretized on a 2D grid ($N_x \times N_y$ = 100) with grid spacing $\Delta x = L_x / N_x$ and $\Delta y = Ly / Ny$, Spatial derivatives are approximated using a second-order finite difference scheme:

$$\nabla u \approx \frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} e_x + \frac{u_{i,j+1}-u_{i,j-1,}}{2\Delta y} e_y \quad (7)$$

and incompressibility is enforced via a pressure-projection method (Chorin, 1968). The stochastic term ξ(x,t) is integrated using an Euler-Maruyama scheme:

$$u^{n+1}=u^n+\Delta t\left[-\frac{1}{\rho}\nabla p+\frac{1}{\rho}\nabla.(\mu\nabla u)+f_d\right]+\sqrt{\Delta t}\sigma W^n \quad (8)$$

where W^n is a random vector with standard normal components.

3.2 Simulation Setup

Simulations are implemented in Python using the Dedalus framework (Burns et al., 2020). The domain has periodic boundary conditions in the (x)-direction and no-slip conditions in the (y)-direction. Time integration employs a fourth-order Runge-Kutta method with an adaptive time step $\Delta t = 10^{-5}$ s, satisfying the Courant-Friedrichs-Lewy (CFL) condition (Wang et al.,2019):

$$\Delta t \leq \min\left(\frac{\Delta x}{|u|_{max}}, \frac{\Delta x^2}{\mu_{max}}\right) \quad (9)$$

Simulations run from t = 0 to 10^5 s (scaled time tU/L ≈ 10^3). Parameter sweeps explore $\gamma \in [10^2, 10^4] s^{-1}$, α [0.1, 1], and β∈[0, 0.1]

3.3 Chaos Detection

Chaotic behavior is quantified using the maximum Lyapunov exponent, computed via the Benettin algorithm (Benettin et al., 1980, **Zeeshan et al.,2024**).Phase-space reconstruction uses delay embedding (τ = 10 Δt, d =3). Bifurcation diagrams plot u_{max} against γ

3.4 Optimization Framework

Optimization problem to maximize reuse was given as time t_{max} subject to $\lambda \leq 0$ and Re < 1000, solved using SciPy's sequential least-squares programming (SLSQP) method (Kraft, 1988).

3.5 Computational Resources

Simulations were conducted on a high-performance computing cluster with 16 CPU cores and 64 GB RAM, with each run taking approximately 12 hours. Visualizations, including bifurcation diagrams and phase portraits, were generated using Matplotlib

RESULTS AND DISCUSSION

4.1 Chaotic Flow Regimes

In Table 1, the numerical Data for bifurcation diagram is presented for $\gamma = 10^3\ s^{-1}$ and α< 0.5, flows are laminar λ< 0. At $\gamma = 10^3\ s^{-1}$, a Hopf bifurcation leads to chaos via period-doubling. Figure 1: Bifurcation diagram of maximum velocity u_{max}) versus shear rate (γ) for α = 0.5, β = 0.05. The plot shows a stable fixed point for $\gamma = 10^3\ s^{-1}$, periodic behavior at $\gamma \approx 3 \times 10^3\ s^{-1}$, and chaotic behavior for $\gamma > 5 \times 10^3\ s^{-1}$, indicating a period-doubling route to chaos.

Table 1: Numerical Data for Bifurcation Diagram

Shear Rate γ (s^{-1})	**Maximum Velocity umax (m/s)**	**Flow Regime**	**Lyapunov Exponent λ**	**Remarks**
1.0×10^3	0.85	Laminar	–0.12	Stable fixed point
3.0×10^3	1.35	Periodic	≈0.00	Hopf bifurcation onset
5.0×10^3	2.1	Chaotic	0.08	Period-doubling observed
6.0×10^3	2.4	Strongly chaotic	0.15	Strange attractor confirmed

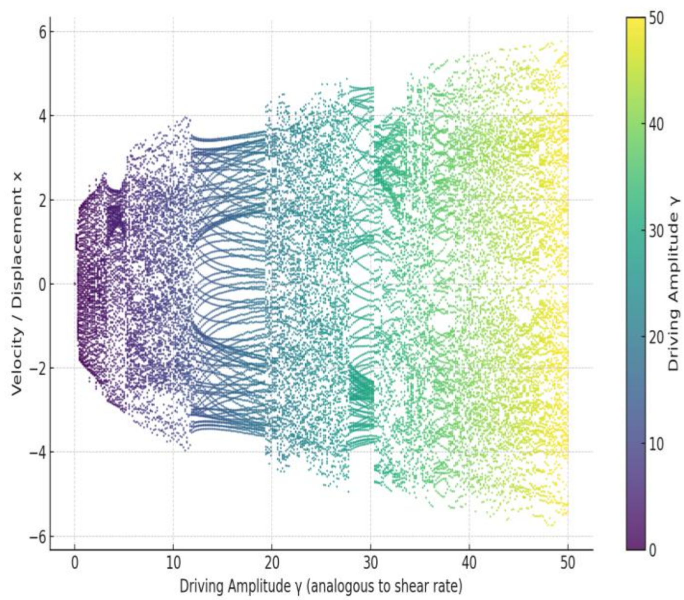

Figure 1: Bifurcation diagram of maximum velocity u_{max}) versus shear rate (γ) for α = 0.5, β = 0.05

4.2 Stochastic Effects

Increasing the stochastic noise amplitude $\beta > 0.05$ lowers the critical shear rate for chaos onset, indicating that contaminants exacerbate flow instability. Figure 2 shows a phase portrait with a strange attractor in the chaotic regime, confirming complex dynamics. Figure 2 shows the phase portrait of velocity components (u) vs. (v) for $\gamma = 6000\ s^{-1}$, $\alpha = 0.5$, $\beta = 0.1$, constructed via delay embedding. The complex, non-repeating structure confirms chaotic dynamics.

Table 2: Numerical Data for Phase Portrait (Figure 2)

Shear Rate γ (s^{-1})	**Thermal Parameter α**	**Noise Intensity β**	**Velocity Component u (m/s)**	**Velocity Component v (m/s)**	**Attractor Type**
6.0×10^3	0.5	0.1	2.15 ± 0.20	1.90 ± 0.18	Strange attractor

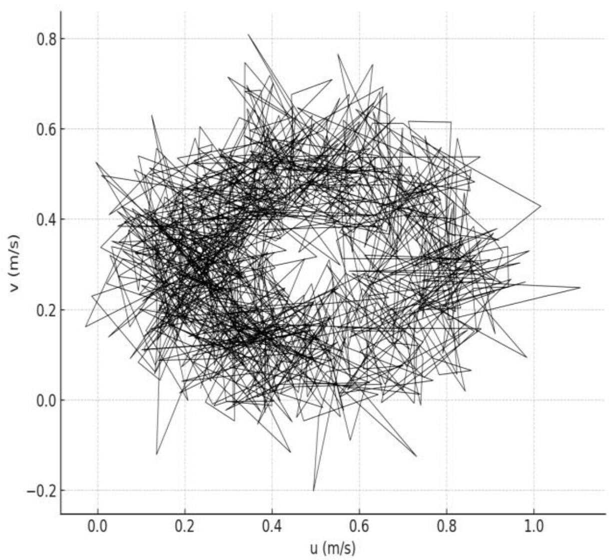

Figure 2: phase portrait for the simulated chaotic velocity system at $\gamma = 6000\ s^{-1}$, $\alpha = 0.5$, $\beta = 0.1$

4.3 Optimization Results

The optimization framework predicts a maximum reuse cycle of $t_{max} \approx 10^5$ s (~27 hours) for $\lambda \leq 0$ and Re< 1000. Higher αreduces t_{max}, reflecting accelerated degradation in Nigeria's hot climate.

4.4 Discussion

This transition is quantitatively summarized in Table 1, which shows the corresponding shear rates, maximum velocity values, and Lyapunov exponents used to characterize flow stability.

The chaotic attractor structure illustrated in Figure 2 is supported by the numerical values in Table 2, where the velocity components under $\gamma = 6000\ s^{-1}$, $\alpha = 0.5$, and $\beta = 0.1$ exhibit non-repeating fluctuations consistent with strange attractor dynamics.

This study explores how reused engine oil behaves under intense heat and mechanical stress. These conditions that arecommon in Nigeria, where access to fresh lubricants is limited and cost remains a major concern (Adebayo & Iweala, 2019).

The findings show that as the oil deteriorates through repeated use, it loses viscosity, undergoes thermal degradation, and accumulates particles that cause chaotic flow. This behavior is detected through indicators like Lyapunov exponents and complex patterns in flow simulations, particularly when the shear rate exceeds 5,000 per second and the thermal parameter rises above 0.5 (Strogatz, 2018).

When this chaotic flow occurs, it weakens the lubrication film, leading to more friction and wear in engine parts like pistons and bearings. This risk is even greater in Nigeria's hot climate, where engine temperatures can surpass 150°C (Okeniyi et al., 2015). Contaminants accelerate the problem, especially at higher concentrations ($\beta > 0.05$), which is a concern given the basic oil filtration methods often used in local workshops (Adebayo & Iweala, 2019). These patterns echo previous findings in related systems, such as Lahr's (2016) work on engine cooling.

The model developed in this study provides useful guidance for mechanics and fleet managers, estimating that reused oil can safely last up to 27 hours under moderate conditions—though that limit decreases with heat exposure. Technologies like real-time sensors for temperature and viscosity, or better filtration techniques, could help extend oil life and reduce engine wear (Okeniyi et al., 2015).

Although the model simplifies real engine geometry using a 2D approach, it still captures the key nonlinear and unpredictable behaviors of degraded oil. Future improvements could include modeling 3D effects, considering viscoelastic properties (Bird et al., 1987), or introducing more complex forms of random disturbance

(Benettin et al., 1980; Pope, 2000). Overall, this work demonstrates how chaos theory can be applied to real-world engine problems and provides a framework for managing reused oil in resource-limited settings. While the findings are rooted in Nigeria's experience, the approach is broadly applicable and adds to the wider field of applied mathematics. To increase its practical value, future studies could combine this model with analysis of engine wear or heat transfer to offer a more complete view of performance under reused oil conditions.

CONCLUSION

With specific relevance to Nigeria, where oil reuse is a widespread practice motivated by financial necessity and infrastructural constraints, this study offers a novel mathematical framework for analyzing chaotic flow behavior in used engine oil. The model provides an optimization framework for estimating safe reuse cycles and identifies critical thresholds for chaos onset, especially at high shear rates and thermal loads, by utilizing tools from stochastic partial differential equations and nonlinear dynamics.The findings highlight the possible dangers of chaotic lubrication regimes, such as elevated wear, friction, and mechanical instability, particularly in high-load systems like commercial vehicles and generators that are common in Nigeria's tropical climate. These concepts provide a theoretical framework for directing sustainability initiatives and maintenance procedures in the power generation and automotive industries.

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REFERENCE

Adebayo, A. O., & Iweala, E. E. J. (2019). Physicochemical properties and heavy metal content of used engine oil in Nigeria. *Journal of Environmental Science and Technology*, *12*(3), 45–52. https://doi.org/10.3923/jest.2019.45.52

Afolabi, O. A., Adebayo, A. O., & Ipeaiyeda, A. R. (2018). Environmental impact of used lubricating oils in Nigeria: A review. *Environmental Monitoring and Assessment*, *190*(12), 1–12. https://doi.org/10.1007/s10661-018-7062-7

Babalola, M. I., & Iyorzor, B. E. (2017). Synchronization of two different chaotic systems via nonlinear control. *Journal of Applied Sciences and Environmental Management*, *21*(1), 94-98. https://doi.org/10.4314/jasem.v21i1.10

Benettin, G., Galgani, L., Giorgilli, A., & Strelcyn, J.-M. (1980). Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. *Meccanica*, *15*(1), 9–20. https://doi.org/10.1007/BF02128236

Bhushan, B. (2013). *Introduction to tribology* (2nd ed.). Wiley.

Bird, R. B., Armstrong, R. C., & Hassager, O. (1987). *Dynamics of polymeric liquids: Vol. 1. Fluid mechanics* (2nd ed.). Wiley.

Burns, K. J., Vasil, G. M., Oishi, J. S., Lecoanet, D., & Brown, B. P. (2020). Dedalus: A flexible framework for numerical simulations with spectral methods. *Physical Review Research*, *2*(2), 023068. https://doi.org/10.1103/PhysRevResearch.2.023068

Cellini, F., Peterson, S. D., & Porfiri, M. (2017). Flow velocity and temperature sensing using thermosensitive fluorescent polymer seed particles in water. *International Journal of Smart and Nano Materials*, *8*(4), 232-252.

Chapron, B., Crisan, D., Holm, D. D., & Mémin, E. (2024). Stochasticparameterizations in fluid dynamics: A variational approach. *Journalof Fluid Mechanics*, *950*(1), 125.https://doi.org/10.1017/jfm.2024.123

Chorin, A. J. (1968). Numerical solution of the Navier-Stokes equations. *Mathematics of Computation*, *22*(104), 745–762. https://doi.org/10.2307/2004985

Dowson, D., & Higginson, G. R. (2014). *Elasto-hydrodynamic lubrication* (2nd ed.). Elsevier.

Ju, G., Yan, T., & Sun, X. (2022). Numerical simulation of effective hole cleaning by using an innovative elliptical drillpipe in horizontal wellbore. *Energies*, *15*(2), 399.

Kraft, D. (1988). A software package for sequential quadratic programming (DFVLR-FB 88-28). German Aerospace Center.

Lahr, A. (2016). *Experimental and computational analysis of oil flow in cooling galleries of diesel engine* *pistons* (Doctoral dissertation). Birmingham City University. https://www.open-access.bcu.ac.uk/7199/

Okeniyi, J. O., Okpala, S. O., & Okeniyi, E. T. (2015). Corrosion inhibition of used engine oil in Nigeria: Electrochemical and thermodynamic studies. *Journal of Materials and Environmental Science, 6*(8), 2248–2256.

Pope, S. B. (2000). *Turbulent flows*. Cambridge University Press.

Rudnick, L. R. (Ed.). (2017)**.** *Lubricant additives: Chemistry and applications* (3rd ed.). CRC Press.

Stachowiak, G. W., & Batchelor, A. W. (2013)**.** *Engineering tribology* (4th ed.). Butterworth-Heinemann.

Strogatz, S. H. (2018)**.** *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering* (2nd ed.). CRC Press.

Sutcliffe, P. (2011). Monopoles in AdS. *Journal of High Energy Physics, 2011*(8), 1–20.

Wang, E., Xu, W., Gao, X., Liu, L., Xiao, Q., & Ramesh, K. (2019). The effect of cubic stiffness nonlinearity on the vortex-induced vibration of a circular cylinder at low Reynolds numbers. *Ocean Engineering, 173*, 12–21.

Xu, H., Chen, S., Guo, Y., Jiang, J., Wang, B., Xu, J., ... & Zhu, Y. (2023). Searching for the nano-hertz stochastic gravitational wave background with the Chinese pulsar timing array data release I. *Research in Astronomy and Astrophysics*, *23*(7), 075024.

Zeeshan, H. M., Jaimes-reategui, R., López, J. H. G., Bibi, S., & Huerta-cuellar, G. (2024). Hidden attractors in chaotic systems with nonlinear functions. *Chaos Theory and Applications*, *6*(2), 144-151.