**On Singh’s dressed epsilon perspective of multigroup**

**Abstract:**

This paper investigates some aspects of multigroups from Singh's perspective. It introduces a structured approach to analyzing the aspects of multigroups presented in this paper using dressed epsilon notation. We begin by defining the hierarchical decomposition of multisets, establishing that each -level reference set in the hierarchical decomposition of a multiset over a group is itself a subgroup. We then present a fundamental characterization of multigroups by proving that a multiset is a multigroup over a set if and only if . Additionally, we define the sets and and prove that both are subgroups of using Singh’s dressed epsilon notation. Our work further investigates the algebraic properties of multigroups and establishes criteria for commutativity. We also demonstrate that while the intersection of two multigroups is always a multigroup, their union does not necessarily inherit this structure. The concept of submultigroup is introduced to formalize the relationship between two multigroups. Finally, we establish the equivalence between certain multigroup properties, such as the symmetry of multisets based on product of elements and conjugate conditions.

**Keywords**: Dressed epsilon, Multigroup Operations, Singh, Submultigroup

**AMS Classifcation:** 03E20, 06D72, 03E72, 20N25

**1 INTRODUCTION**

The concept of groups forms a fundamental pillar in abstract algebra, with applications spanning fields such as mathematics, computer science and physics. Classical group theory primarily deals with sets and binary operations that satisfy closure, associativity, the existence of an identity element, and the presence of an inverse. However, there is a growing interest in generalizing these notions to accommodate multisets, leading to the concept of multigroups (Baumslag and Chandler, 1968).

Singh introduced the notion of a multigroup as a natural extension of the classical group, utilizing the dressed epsilon notation to capture the multiplicity of elements within the group structure. The dressed epsilon notation provides a precise way to express that an element belongs to a multiset with a certain multiplicity, thereby facilitating the extension of group operations to multisets. The motivation behind studying multigroups lies in their potential to model complex systems where redundancy and repetition are inherent, such as in computational and combinatorial contexts. (Singh, 2006 **and** Singh et al., 2008).

In this paper, we explore the fundamental properties and structure of multigroups within the framework of group theory. We begin by establishing the hierarchical decomposition of multisets, providing a systematic way to organize elements based on their multiplicity levels. This decomposition is crucial for understanding how multigroups can be decomposed into structured subgroups (Singh and Isah, 2016).

We then delve into the properties of multigroups, presenting necessary and sufficient conditions for a multiset to qualify as a multigroup. We investigate the product and inverse conditions that characterize multigroups and examine how these properties extend from traditional group axioms. Furthermore, we explore the algebraic relationships between multigroups, including the commutative properties of multigroup multiplication, and provide conditions under which the product of two multigroups remains a multigroup. We also examine intersection and union operations on multigroups, highlighting scenarios where these operations preserve the multigroup structure.​(Ejegwa and Ibrahim, 2020).

Through this study, we aim to enrich the theoretical foundation of multigroups by identifying their structural characteristics, algebraic properties, and potential applications. Our results not only bridge the gap between classical group theory and multiset theory but also open avenues for applying multigroup concepts to real-world problems where element repetition plays a significant role (Peter, et al., 2025).

Recent studies have further enriched the understanding of multigroups. Nazmul et al. (2013) introduced foundational concepts related to multigroups derived from multisets. Awolola and Ibrahim (2016) explored various properties of multigroups, while Awolola and Ejegwa (2017) examined the order of elements within these structures. Awolola (2019) investigated cyclic multigroup families, shedding light on their structural characteristics. Additionally, Ibrahim et al. (2016) provided a comprehensive overview of multigroup theory and its potential applications.

**2 METHODOLOGY**

1. This study employs a theoretical approach to investigating the properties and structures of multigroups within the framework of multiset theory.
2. The research methodology encompasses the following key steps. Previous works on multigroups are analyzed.
3. The study established a theoretical framework for multigroups. This included defining multigroups in the context of multisets and formulating precise definitions and notations to capture the multiplicity of elements within these structures.
4. A series of propositions were formulated to characterize the algebraic properties of multigroups. Each proposition was accompanied by rigorous mathematical proofs to validate the conditions under which a multiset qualifies as a multigroup.
5. The study investigated various operations within multigroups, including multiplication, inversion, intersection, and union. The closure properties of these operations were examined to determine how they affect the multigroup structure.
6. Specialized substructures within multigroups, were defined and analyzed. The study explored the conditions under which these substructures form subgroups and their significance in the broader context of multigroup theory.

**3 PRELIMINARY DEFINITIONS**

**Definition 3.1 (Multiset)** A multiset over a domain set is a collection of elements of with repetitions allowed. The set is called the ground or generic set of the class of all multisets containing elements from .

**Definition 3.2 (Union)** Suppose and are two multisets over a ground set , then is the multiset defined by .

**Definition 3.3 (Intersection)** Suppose and are two multisets over a ground set , then is the multisetdefined by .

**Definition 3.4** Let and be multigroups over a group . The product is defined such that for any element , there exist elements and such that where denotes the group operation..​

**4 RESULTS AND DISCUSSION**

**Definition 4.1 (Multigroup)** Let be a group. A multiset over is said to be a multigroup over if it satisfies the following two conditions:

1. (Multiplication condition)
2. (Inverse condition)

**Example 4.2**

For example, consider the cyclic group of order 4 be the cyclic group of order 4, where . Let the multiset be a multiset over . The membership conditions are given as follows:

Multiplication conditions:

For all , we verify that the multiplication condition

holds:

Inversion condition

For all , we verify:

Since both conditions hold, we conclude that is a multigroup over .

**Proposition 4.3**

Let be a multiset. Then is a multigroup over a set if and only if

*Proof*

Assume is a multigroup over , By definition of a multigroup, we have the following two conditions:

1. Multiplication condition: If and then .
2. Inverse condition: If , then .

Since is a multigroup, for any , we have: (by the inverse) condition. and (by the multiplication condition). Thus, as required.

Conversely, assume that for any such that and , we have . We need to show that satisfies the multiplication and inverse conditions of a multigroup. Now take . Then, , where is the identity element . Hence, . Thus, the inverse condition is satisfied.

Now, for any , we know from the hypothesis, that . Sincethe group operation is closeand the inverse condition holds, then multiplication condition also holds.

**Definition 4.4 (Hierarchical decomposition of multisets)** Let be a mutiset over a set , then the set is called -level reference of where is the position of the reference set when all the reference sets (the empty set inclusive) are arranged in a descending order using the non-proper containment relation . In this case, the set for each is known as an *-reference set*.

**Proposition 4.5**

*Let M be a multiset over a group X. The r-level reference sets in the hierarchical decomposition of multiset M are subgroups of X.*

*Proof*

Since is a group, we need to show that for any , the element also belongs to . Since , it means that and . By the definition of a multigroup

Since both and appear at least times in , we have that Therefore, . Thus,
. This shows that .

**Defnition 4.6** Let be a multigroup over a group . Define and as

and

**Proposition 4.7** Let be a multigroups over a group then and are submultigroups of .

*Proof*

A is a subgroup, the identity element of the group is also in with some multiplicity . By the definition of , for any , and imply . Hence, the identity element satisfies the condition and is in .

Take any . Then and imply . Since is a multigroup, the product must also be in with multiplicity at least which in this case is equal to since Thus, . Therefore, is a subgroup of

Since is a multigroup and is the identity, must be present in with a positive multiplicity. Hence,

Take any Then Since is a multigroup, the product must be in with multiplicity at least which is strictly positive. Therefore, . Therefore, is a subgroup of .

**Proposition 4.8** A multiset is a multigroup over a group if and only if the following properties are satisfied:

1. *;*

*Proof*

Assume that is a multigroup over a group . Now for any such that , . This implies that the product of any two elements from is still in with at least the minimum multiplicity. Therefore, .

Also, for any such that , . This means that the inverse of any element from is also in with the same multiplicity, hence,

Conversely, since the first condition states that the product of any two elements from is still in it satisfies the multiplicity condition of a multigroup. The second condition states that the inverse of any element in is also in , satisfying the inverse ondition of a multigroup.

**Proposition 4.9** A multiset is a multigroup over a group if and only if the following properties are satisfied .

*Proof*

Assume that is a multigroup over a group . By the definition of a multigroup, For any such that , . This implies that the product of any element from and the inverse of any other element from is still in with at least the minimum multiplicity. Therefore, .

Conversely, Assume that the given property holds: that is, . Since this condition states that the product of any element form is still in , it satisfies the multiplication condition of a multigroup. Furthermore, the inverse condition of a multigroup states that for any the inverse , which follows from the fact that the set is closed under taking inverses.

**Proposition 4.10** Let and be multigroups over a group . Then is a multigroup over if and only if

*Proof*

Assume is a multigroup, it must satisfy the multiplication condition and the invese condition. By the multiplication condition for multigroups: For any such that , . Similarly, since is a multigroup under the same conditions, . Since for any then

Conversely, assume If the multigroup product is commutative, then both multiplication and inverse conditions hold symmetrically for . Thus, is closed under multiplication and inverse conditions. Hence, is a multigroup over

**Proposition 4.11** Let and be multigroups over a group . Then is a multigroup over

*Proof*

For any and since and , and and , by the multiplication condition for multigroups,

Thus:

Hence, the multiplication condition holds. For any , since and by the inverse condition for multigroups, and . Thus

Hence is a multigroup over .

Next, we show the case is not true for union. Consider the cyclic group where is the identity element and the group operation is as follows:

Consider the multigroups and over . With the above information we are now ready to state and prove the next proposition.

**Remark 4.12** It is worth noting here that is not necessarily a multigroup. To see this consider the multigroups and over the group as in the above proposition. Consider their union . For the union to be a multigroup it must satisfy the multiplication condition: For any and

However, consider and from . and . However , . Hence,

Thus, the multiplication condition is not satisfied.

**Definition 4.13** Let and be multigroups over a group . Then is a submultigroup of denoted if .

**Example 4.14** Let be the cyclic group of order 4: under the multiplication modulo 4, where is the identity element. Consider the multigroups and . Since the multiplicity of every element in is less or equal to that in , then . Moreover, is a multigroup based on definition.

In Propositions 4.15 we present the results that establish the equivalence between the properties of multigroup based on conjugation of elements.

**Proposition 4.15** Let be a multiset over a set . Then the following assertions are equivalent:

*Proof:*

To prove (i) (ii):

Assume . We know that . Putting in place of in (i) we get:

This simplifies to

To prove (ii) (i): Assume . The statement essentially says that the multiplicity of an element remains the same when conjugated by any element , i.e., has the same multiplicity as in the multiset . In particular, if and have the same multiplicity, it suggests that the multiset structure respects some form of symmetry under element swapping. Hence,

In Propositions 4.16 we present the results that establish the equivalence between the properties of multigroup based on product of their elements.

**Proposition 4.16** Let be a multiset over a set . Then the following assertions are equivalent:

*Proof:*

1. (ii)

Assume . Consider any multiset over

 By the definition of the product of two multisets:

By hypothesis, the multiplicity of the product and is the same, it follows that:

1. (i)

Assume . Choose an element and form a singleton multiset . Now

This means

Hence,

**5 CONCLUSION**

In this study, we have extended classical group theory into the realm of multisets, introducing and formalizing the concept of multigroups. By utilizing Singh's dressed epsilon notation, which denotes that an element belongs to a multiset at least once, we have analyzed multisets and some-level reference sets and their intrinsic subgroup properties.​

Through the propositions and proofs presented, we established the foundational properties of multigroups, showing the conditions under which a multiset qualifies as a multigroup. Our exploration of multigroup operations, including multiplication, inversion, intersection, and union, revealed the nuanced ways in which these operations preserve or alter multigroup structures.

The incorporation of the dressed epsilon notation has been instrumental in extending the applicability of group theory to multisets, allowing for flexible expression multiplicities objects. This advancement not only bridges the gap between traditional group theory and multiset theory but also opens avenues for practical applications in areas where multiblicities of objects is significant. This work paves the way for future research into the applications of multigroups in computational mathematics, data analysis, and other related fields.

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